TAUTOCHRONES AND BRACHISTOCHRONES.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, February 27, 1909.)

IN the simplest case of a particle acted upon by gravity, the tautochrone problem, solved by Huygens in 1673, and the brachistochrone problem, solved by Jean Bernoulli in 1697, give rise to the same curves, namely, cycloids with horizontal bases and concavity upwards. In the case of a general field of force the two problems lead to distinct systems of curves. Their differential equations, each of the third order, are given in the first section of this note. In § 2 it is shown that the only force besides gravity for which the two systems coincide is the central force varying directly as the distance from the origin. If the force generating the brachistochrones is not required to be the same as that generating the tautochrones, then a third case of duplication is possible (§ 4). Incidentally, a class of forces involving eight parameters and related to infinitesimal collineations presents itself (§ 3); they are the only fields of force for which every straight line of the plane is a tautochrone.

§ 1. *General Equations.*

We consider a particle of unit mass moving in the plane under a force whose rectangular components are $\phi(x, y)$, $\psi(x, y)$. With reference to an arbitrary curve the normal and tangential components are

$$N = \frac{\psi - y'\phi}{\sqrt{1+y'^2}}, \quad T = \frac{\phi + y'\psi}{\sqrt{1+y'^2}}. \tag{1}$$

The condition for a tautochrone is that the motion along the curve be harmonic, that is,

$$T = \kappa(s - s_0), \tag{2}$$

where κ is a constant* and $s - s_0$ denotes the arc reckoned

* For an *actual* tautochrone κ must be negative. The differential equation (3) applies also when κ is positive. Such curves may be termed *virtual* tautochrones. They are the actual tautochrones of the reversed force. A similar remark applies to trajectories, brachistochrones, and catenaries.

from a fixed point of the curve, the center of the tautochronous motion. Differentiating twice with respect to s, we have, as the equation of all the tautochrones,

$$T_{ss} = 0. \tag{3}$$

The brachistochrones may be found most readily from Euler's theorem on pressure, which gives

$$\frac{v^2}{r} = -N, \tag{4}$$

where v is the speed and r the radius of curvature.

Differentiating both sides with respect to s, and remembering that

$$vv_s = \frac{dv}{dt} = T, \tag{5}$$

we find the equation of the family of brachistochrones to be *

$$2T + rN_s + Nr_s = 0. \tag{6}$$

Equations (3) and (6) when expanded are seen to involve third derivatives of y with respect to x, so that each defines a triply infinite system of curves.

To obtain the results in more explicit form we introduce an auxiliary vector, completely defined by the given field of force, namely, the space derivative of the force (considered as a vector). The rectangular components of the new vector are

$$\phi_s = \frac{\phi_x + y'\phi_y}{\sqrt{1+y'^2}}, \quad \psi_s = \frac{\psi_x + y'\psi_y}{\sqrt{1+y'^2}}, \tag{7}$$

and its normal and tangential components are

$$\begin{aligned} \mathfrak{N} &= \frac{\psi_s - y'\phi_s}{\sqrt{1+y'^2}} = \frac{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2}{1+y'^2}, \\ \mathfrak{T} &= \frac{\phi_s + y'\psi_s}{\sqrt{1+y'^2}} = \frac{\phi_x + (\phi_y + \psi_x)y' + \psi_y y'^2}{1+y'^2}. \end{aligned} \tag{8}$$

* This result, like Euler's theorem, applies only to conservative forces. Apparently the theory of brachistochrones for non-conservative fields has not been investigated. The system in this case consists in fact of ∞^4 instead of merely ∞^3 curves. Since the tautochrone system always contains ∞^3 curves, such forces need not be considered in the problems of § 2 and § 4.

The following noteworthy relations may be derived without difficulty:

$$N_s = \mathfrak{N} - \frac{T}{r}, \qquad T_s = \mathfrak{T} + \frac{N}{r}. \tag{9}$$

We require finally the space derivatives of $\mathfrak{N}$ and $\mathfrak{T}$, which may be written in the form

$$\mathfrak{N}_s = \mathfrak{N}_1 + \frac{\mathfrak{N}_2}{r}, \qquad \mathfrak{T}_s = \mathfrak{T}_1 + \frac{\mathfrak{T}_2}{r}, \tag{10}$$

where

$$\begin{aligned}
\mathfrak{N}_1 &= \frac{\psi_{xx} + (2\psi_{xy} - \phi_{xx})y' + (\psi_{yy} - 2\phi_{xy})y'^2 - \phi_{yy}y'^3}{(1+y'^2)^{\frac{3}{2}}}, \\
\mathfrak{N}_2 &= \frac{\psi_y - \phi_x - 2(\phi_y + \psi_x)y' + (\phi_x - \psi_y)y'^2}{1+y'^2}, \\
\mathfrak{T}_1 &= \frac{\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3}{(1+y'^2)^{\frac{3}{2}}}, \\
\mathfrak{T}_2 &= \frac{\phi_y + \psi_x + 2(\psi_y - \phi_x)y' - (\phi_y + \psi_x)y'^2}{1+y'^2}.
\end{aligned} \tag{11}$$

The functions ϕ, ψ depend only on the position of the particle; the auxiliary functions N, T, $\mathfrak{N}$, $\mathfrak{T}$, $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{T}_1$, $\mathfrak{T}_2$, defined above, depend also upon the direction of motion; finally, N_s, T_s, $\mathfrak{N}_s$, $\mathfrak{T}_s$ depend upon the curvature of the path.

Making use of (9) and (10), we may reduce our equations (3) and (6) to explicit form, and obtain this result:

The general equation of the system of tautochrones is

$$Nr_s = \mathfrak{T}_1 r^2 + (\mathfrak{T}_2 + \mathfrak{N})r - T, \tag{I}$$

while that of the system of brachistochrones is

$$Nr_s = -\mathfrak{N}r - T. \tag{II}$$

From these equations we may deduce the general geometric properties of the two systems. We note merely that for tautochrones r_s (the rate of variation of the radius of curvature per unit of arc) is a quadratic function of r, while for brachistochrones it is a linear function of r. The results (I) and (II)

are of course differential equations of the third order with respect to y as a function of x. The usual notation may be obtained by substituting

$$r = \frac{(1+y'^2)^{\frac{3}{2}}}{y''}, \quad r_s = \frac{3y'y''^2 - (1+y'^2)y'''}{y''^2}, \tag{12}$$

but the above intrinsic forms are more convenient in the following.

We note in passing that for any field of force the equation

$$Nr_s = -\mathfrak{R}r + mT, \tag{13}$$

involving an arbitrary parameter m, includes the following four distinct systems of curves of physical interest: brachistochrones when $m = -1$, trajectories when $m = 3$, catenaries when $m = 2$, and velocity curves when $m = 1$. For all values of m the curves may be obtained by requiring that the pressure shall vary directly as the normal component of the external force. No cases of duplication, beyond those already mentioned, are found by comparing these systems with tautochrones or with one another.

§ 2. *The Duplication Problem for a Single Field.*

We proceed to find the forces for which the brachistochrone system coincides with the tautochrone system. The conditions expressing the equivalence of equations (I) and (II) are

$$\mathfrak{T}_1 = 0, \tag{14}$$

$$\mathfrak{T}_2 + 2\mathfrak{R} = 0. \tag{15}$$

Expanding the latter, we find

$$\phi_y + 3\psi_x = 0, \quad \psi_y - \phi_x = 0, \quad 3\phi_y + \psi_x = 0. \tag{15'}$$

The solutions are easily found to be

$$\phi = cx + a, \quad \psi = cy + b, \tag{16}$$

where a, b, c are arbitrary constants. No new restrictions are imposed by (14). By a simple change of axes and scale the field (16) may be written in one of these two canonical forms,

$$\phi = 0, \quad \psi = 1, \tag{17}$$

$$\phi = x, \quad \psi = y. \tag{18}$$

The only types of force for which the tautochrones coincide with the brachistochrones are represented by (17) *and* (18). *In the first type the force is constant in direction and intensity* (*gravity*); *in the second type the force is central and varies directly as the distance from the center* (*elastic law*).

The system of curves in the first type consists of cycloids with horizontal bases; in the second type the curves are hypocycloids and epicycloids, including as limiting cases equiangular spirals, all having the origin as center.

§ 3. *A Class of Forces Related to Collineations.*

We now determine those forces for which the right hand member of the general tautochrone equation (I) is linear in r.* The condition for this is $\mathfrak{T}_1 = 0$, which decomposes on expansion into the set of partial differential equations

$$\begin{aligned} &\phi_{xx} = 0, \quad 2\phi_{xy} + \psi_{xx} = 0, \\ &\phi_{yy} + 2\psi_{xy} = 0, \quad \psi_{yy} = 0. \end{aligned} \tag{19}$$

The integration of this set may be carried out without difficulty; but the calculation may be avoided by noticing a simple connection with the equations arising in the determination of infinitesimal collineations. If an infinitesimal transformation, represented symbolically by

$$\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y},$$

is to be projective, then the well known conditions are

$$\begin{aligned} &\eta_{xx} = 0, \quad 2\eta_{xy} - \xi_{xx} = 0, \\ &\eta_{yy} - 2\xi_{xy} = 0, \quad \xi_{yy} = 0. \end{aligned}$$

This is converted into our set (17) by replacing ξ by ψ and η by $-\phi$. Hence the solutions of (17) are

$$\begin{aligned} -\phi &= b + cx + gy + hxy + ky^2, \\ \psi &= a + cx + dy + hx^2 + kxy, \end{aligned} \tag{20}$$

involving eight arbitrary constants, the notation being taken in agreement with Lie-Scheffers, Continuierliche Gruppen, page 24. Our result is then:

* The linear form is equivalent to property I of a system of dynamical trajectories. See *Trans. Amer. Math. Soc.*, vol. 7 (1906), p. 405.

If a system of tautochrones is to have the property that r_s (the rate of variation of the radius of curvature per unit of arc) is a linear function of r (the radius of curvature), then the field of force must be of the class (20). *These fields are characterized by the fact that the related infinitesimal transformation*

$$\psi \frac{\partial}{\partial x} - \phi \frac{\partial}{\partial y} \tag{21}$$

is a collineation.

It follows that the lines of force,

$$\frac{dy}{dx} = \frac{\psi}{\phi}, \tag{22}$$

have for orthogonal trajectories a system of anharmonic curves (W curves). Of course this property is not characteristic, since it describes only the direction of the force, not its intensity.

The forces (20) arise most concretely in response to this problem: In what cases will every straight line be a possible tautochrone?*

For a straight line we have $r = \infty$, and $r_s = 0$. If (I) is to be satisfied by these values, then $\mathfrak{T}_1$, the coefficient of r^2, must vanish. The same result is obtained by reducing (I) to the form

$$y''' = \mathfrak{T}_1 + y' f_1(x, y, y') + y''^2 f_2(x, y, y'), \tag{23}$$

and imposing the condition that $y'' = 0$ shall be a particular integral.

The class of forces (20) *is thus completely characterized by the fact that every straight line of the plane is a possible tautochrone.*

If the force (20) is required to be conservative it is found that the coefficients must satisfy the relations

$$h = 0, \quad k = 0, \quad c + g = 0. \tag{24}$$

The corresponding collineation then becomes an affine transformation leaving areas invariant. The components of the force are of the linear form

$$\phi = -b - ex + cy, \quad \psi = a + cx + dy$$

and the work function is

* Mr. Reddick has shown that in space of three dimensions this problem leads to a class of forces involving twenty parameters; the relation to collineations is therefore peculiar to the plane.

$$W = -bx + ay + cxy - \tfrac{1}{2}ex^2 + \tfrac{1}{2}dy^2. \tag{25}$$

The only conservative forces for which every straight line is a tautochrone are those in which the potential is a quadratic function of x, y.

The lines of force are then the orthogonals of a system of homothetic conics. Such forces are of importance in connection, for example, with the general theory of motion about a position of equilibrium.

§ 4. *The Duplication Problem for Two Fields.*

When will the system of ∞^3 tautochrones connected with one field of force coincide with the system of ∞^3 brachistochrones connected with a second field? We denote the components of the first force by ϕ, ψ, and those of the second force by ϕ^*, ψ^*, so that the differential equations of the two systems considered are

$$Nr_s = \mathfrak{T}_1 r + (\mathfrak{T}_2 + \mathfrak{N})r - T, \tag{26}$$

$$N^* r_s = -\mathfrak{N}^* r - T^*. \tag{27}$$

We observe that a necessary condition for equivalence is $\mathfrak{T}_1 = 0$; hence the first force must be of the type (20). The additional conditions to be satisfied are

$$\frac{T^*}{N^*} = \frac{T}{N}, \quad \frac{\mathfrak{T}_2 + \mathfrak{N}}{N} = -\frac{\mathfrak{N}^*}{N^*}. \tag{28), (29}$$

From the first of these, we find that the ratio of ψ^* to ϕ^* must be equal to the ratio of ψ to ϕ. We may therefore write

$$\phi^* = \rho\phi, \quad \psi^* = \rho\psi, \tag{30}$$

where ρ is an unknown function of x, y. A short calculation shows that, in consequence of (30),

$$\frac{\mathfrak{N}^*}{N^*} = \frac{\mathfrak{N}}{N} + \frac{\sigma_x + y'\sigma_y}{\sqrt{1 + y'^2}} \qquad (\sigma = \log \rho). \tag{31}$$

Equation (29) may then be written in the form

$$\mathfrak{T}_2 + 2\mathfrak{N} + N\sigma_s = 0. \tag{32}$$

Expanding this in powers of y', making use of the known form (20) of ϕ and ψ, and equating coefficients to zero, we find

$$
\begin{aligned}
3c - g + 5hx + ky + \psi\sigma_x &= 0, \\
4(d + e + kx + hy) + \psi\sigma_y - \phi\sigma_x &= 0, \\
3g - c + hx + 5ky - \phi\sigma_y &= 0.
\end{aligned} \tag{32'}
$$

The unknowns are now the eight coefficients involved in (20) and the function σ. Elimination of σ_x and σ_y gives

$$
\begin{aligned}
\phi^2(3g - c + hx + 5ky) + 4\phi\psi(d + e + kx + hy) \\
+ \psi^2(3c - g + 5hx + ky) = 0.
\end{aligned} \tag{33}
$$

By means of the coefficients of the highest powers here involved, namely, x^5 and y^5, we find that h and k must vanish. The last equation thus simplifies to

$$(3g - c)\phi^2 + 4(d + e)\phi\psi + (3c - g)\psi^2 = 0, \tag{33'}$$

where

$$\phi = -b - ex - gy, \quad \psi = a + cx + dy. \tag{34}$$

We now divide the discussion into two cases:

Case 1°. If the coefficients $3g - c$, $4(d + e)$, $3c - g$ appearing in (33′) all vanish, then $c = 0$, $g = 0$, $d = -e$, and we obtain the form (16) of § 2. In fact, under these conditions, equations (32′) show that both σ_x and σ_y will vanish, that is, σ and hence ρ will be a constant, that is, the two fields will coincide.

Case 2°. Under the contrary assumption, we see from (33′) that the ratio of ψ to ϕ must be constant. This means that the force has a constant direction, which we may without loss of generality assume to be the direction of the y-axis. The component ϕ will then vanish, so that, from (34),

$$b = e = g = 0.$$

Equation (33′) then gives $3g - c = 0$, so that c also must vanish. Our force thus becomes

$$\phi = 0, \quad \psi = a + dy.$$

If d vanishes, this becomes the type (17). Otherwise, it may, by a change of axes and scale, be reduced to the form

$$\phi = 0, \quad \psi = y. \tag{35}$$

Substituting these values in (32′) we find that the equations for the unknown function σ are consistent and give

$$\sigma = -4 \log y,$$

so that ρ, the factor for converting the first field into the second field, is y^{-4}. Hence the second field is

$$\phi^* = 0, \quad \psi^* = y^{-3}. \tag{36}$$

In (35) the force varies directly as the distance from the x axis, while in (36) the force varies as the inverse cube of that distance.

The only cases in which a system of tautochrones is also a system of brachistochrones are these three:

1°. *The tautochrones and the brachistochrones of the uniform field* $\phi = 0$, $\psi = 1$ *coincide.*

2°. *The same is true of the elastic field* $\phi = x$, $\psi = y$.

3°. *The tautochrones of the field* $\phi = 0$, $\psi = y$ *coincide with the brachistochrones of the field* $\phi^* = 0$, $\psi^* = y^{-3}$.

All these fields are conservative.

COLUMBIA UNIVERSITY, NEW YORK.

DEGENERATE PENCILS OF QUADRICS CONNECTED WITH $\Gamma^{n+2}_{n+4,n}$ CONFIGURATIONS.

BY DR. W. B. CARVER.

(Read before the American Mathematical Society, December 31, 1908.)

IN a previous paper * the author discussed a certain pencil of quadric spreads associated with the configuration $\Gamma^{n+2}_{n+4,n}$ in space of n dimensions. The $\Gamma^{n+2}_{n+4,n}$ contains $n+4$ configurations $\Gamma^{n+1}_{n+3,n}$, and with each of these is associated a quadric spread with respect to which its points and S_{n-1}'s are poles and polars. In the case of a *proper* $\Gamma^{n+1}_{n+3,n}$, *i. e.*, one whose points, lines, planes, $\cdots$, and S_{n-1}'s are all distinct, it is evident that the associated quadric spread cannot degenerate into a cone.† Hence, for a proper $\Gamma^{n+2}_{n+4,n}$, the individual spreads of the associated pencil cannot be degenerate; but the question naturally arises as to whether the pencil itself, or the quartic $(n-2)$-way spread through which the quadrics all pass, may be degenerate. It is the object of this paper to answer this question for the cases $n = 1$, 2, and 3.

* "The quadric spreads connected with the configuration $\Gamma^{n+2}_{n+4,n}$," *Amer. Jour. of Mathematics*, vol. 31, pp. 1–17 (January, 1909).

† That is, if the quadric spread be represented by a quadratic equation in $n+1$ homogeneous variables, the discriminant of this equation must not vanish.

We will consider the three-dimensional case first, as affording the best illustration of the method of treatment. We have a pencil of seven quadric surfaces associated with a $\Gamma^5_{7,2}$ configuration. It was shown in the author's previous paper that the pencil of quadrics could be first chosen arbitrarily, and that the configuration was then formally determined. The analytic procedure there described * may be used here. We may choose a pencil of quadric surfaces of any one of the thirteen well-known types,† and see whether the configuration then determined is or is not degenerate.

For example, consider a pencil of quadrics for which the common quartic curve consists of a conic and two lines not intersecting on the conic (type 8 in the Clebsch-Lindemann classification). Seven surfaces of such a pencil may be represented with perfect generality by the equations

$$x_2x_3 + x_3x_4 + x_4x_2 + \lambda_i x_1x_4 = 0 \quad (i = 1, 2, \cdots, 7).$$

Let the point 123 of the configuration be a_1, a_2, a_3. Then the plane 14567 will be

$$\lambda_1 a_4 x_1 + (a_3 + a_4)x_2 + (a_2 + a_4)x_3 + (\lambda_1 a_1 + a_2 + a_3)x_4 = 0;$$

the point 234 will be

$$\lambda_1\lambda_4 a_1 + 2(\lambda_1 - \lambda_4)a_4,\ \lambda_4^2 a_2 + \lambda_4(\lambda_4 - \lambda_1)a_4,$$
$$\lambda_4^2 a_3 + \lambda_4(\lambda_4 - \lambda_1)a_4,\ \lambda_1\lambda_4 a_4;$$

the point 345 will be

$$\lambda_2\lambda_3\lambda_4\lambda_1 a_1 + 2\,[\lambda_2\lambda_5(\lambda_1 - \lambda_4) + \lambda_1\lambda_4(\lambda_2 - \lambda_5)]a_4,$$
$$\lambda_4^2\lambda_5^2 a_2 + \lambda_4\lambda_5(\lambda_4\lambda_5 - \lambda_1\lambda_2)a_4,$$
$$\lambda_4^2\lambda_5^2 a_3 + \lambda_4\lambda_5(\lambda_4\lambda_5 - \lambda_1\lambda_2)a_4,$$
$$\lambda_2\lambda_5\lambda_1\lambda_4 a_4;$$

and the plane 12367 will be

$$\Sigma_2 S_2 a_4 x_1 + \Sigma_2^2(a_3 + a_4)x_2 + \Sigma_2^2(a_2 + a_4)x_3$$
$$+ [\Sigma_2 S_1 a_1 + \Sigma_2^2(a_2 + a_3) + 2(\Sigma_2^2 + S_3\Sigma_1 - S_2\Sigma_2)a_4]x_4 = 0,$$

where the Σ's and S's are the ordinary symmetric functions respectively of λ_4, λ_5 and λ_1, λ_2, λ_3. Since the point 123 must lie in the plane 12367, we have the relation

* Loc. cit., p. 5.

† Clebsch-Lindemann, Vorlesungen über Geometrie, vol. II, p. 215.

$$\Sigma_2 S_3 a_1 a_4 + \Sigma_2^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1 S_3 - \Sigma_2 S_2) a_4^2 = 0.$$

Similarly, the coordinates of the point 123 must satisfy the conditions

$$\Sigma_2' S_3 a_1 a_4 + \Sigma_2'^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1' S_3 - \Sigma_2' S_2) a_4^2 = 0$$

and

$$\Sigma_2'' S_3 a_1 a_4 + \Sigma_2''^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1'' S_3 - \Sigma_2'' S_2) a_4^2 = 0,$$

where the Σ''s and Σ'''s are symmetric functions respectively of λ_5, λ_6 and λ_6, λ_7.

Each of the 35 points of our configuration must satisfy such a set of three equations. These equations are linear and homogeneous in the quantities $a_1 a_4$, $a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4$, and a_4^2. The determinant of the coefficients cannot vanish; for it reduces to

$$\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_5^2 \lambda_6^2 (\lambda_6 - \lambda_4)(\lambda_4 - \lambda_7)(\lambda_7 - \lambda_5)$$

and no λ can be zero (for this would give a cone) and no two λ's can be equal. Hence the conditions could only be satisfied if

$$a_1 a_4 = 0, \quad a_4^2 + a_2 a_3 + a_3 a_4 + a_2 a_4 = 0, \quad a_4^2 = 0.$$

But this would make the point 123 lie on one of the two lines which make up part of the quartic curve, and *the same would be true for all the points of the configuration.* Hence we cannot have a proper configuration in this case.

By a similar treatment in the other cases, we obtain the following results:

Proper configurations exist for the types of pencils 1–5, and do not exist for types 6–13. For the general case, type 1, there are eight configurations for any given pencil of seven quadrics;* but for types 2–5 there are fewer configurations for a given pencil, because some of the solutions are absorbed by the special points or lines.

We find a more surprising situation, perhaps, in the plane case. Here we have five types of pencils of conics:

1. Four distinct points of intersection;

2. Two distinct points of intersection, and contact of first order at a third point;

3. Contact of first order at two distinct points;

* Author's paper, loc. cit., p. 6.

4. One point of intersection, and contact of second order at a second point;

5. Contact of third order.

Proper configurations exist for types 1 and 4, and do not exist for types 2, 3, and 5. For a given pencil of six conics there are four configurations for type 1, but only one for type 4.

For the one-dimensional case, we have the configuration $\Gamma^3_{5,1}$ on a line, and the pencil of quadrics is simply five pairs of points in involution.* We have two types, viz., the fixed points of the involution are distinct or they are coincident. Proper configurations exist for both cases. Given the five point pairs, there are two configurations determined in the first case, but only one in the second. This one-dimensional case is readily seen if the whole figure is projected upon a conic.

ON THE USE OF n-FOLD RIEMANN SPACES IN APPLIED MATHEMATICS.

BY PROFESSOR JAMES McMAHON.

THE object of this article is to show that the conception of a Riemann surface has important physical bearings, and to indicate in a general way what kind of physical problems have been solved or may be solvable by the use of such n-fold surfaces or analogous manifold regions in three dimensions. The most recent work in this line constitutes the highest point yet reached in the application of modern function theory to physical problems. It is very noteworthy that a theory which was developed by following out purely intellectual relations, without any reference to the world of sense, should afterwards find unexpected applications and correspondences in the physical universe.

The conceptions of multiform functions, and of multiple spaces in which such functions are made uniform, furnish elegant solutions of some important problems in the theories of potential, electricity, light, sound, heat, and fluid motion. To give greater clearness to what follows, it may be well to take a simple illustration of a three-valued potential function in two dimensions, and show how to make it one-valued on a three-fold Riemann surface. Let (ρ, θ) be the polar coordinates of a

*Author's paper, loc. cit., p. 5, footnote.

point in a plane, and consider the function $\rho^{\frac{1}{3}} \cos \frac{1}{3}\theta$, which satisfies Laplace's equation, and has in general three real and distinct values at the point (ρ, θ), the second and third values being obtained by changing θ into $\theta + 2\pi$ and $\theta + 4\pi$ in succession. On making successive circuits around the origin, the first value changes continuously into the second, the second into the third, and the third into the first. It will be seen that a pair of values are equal when $\theta = 2\pi$, another pair when $\theta = 4\pi$, and the other pair when $\theta = 6\pi$. Now let us imagine the plane covered over by two other plane sheets in such a way that a revolving radius vector shall trace out the first sheet from $\theta = 0$ to $\theta = 2\pi$, and then by continuous motion trace out the second sheet from $\theta = 2\pi$ to $\theta = 4\pi$, and the third sheet from $\theta = 4\pi$ to $\theta = 6\pi$, passing then into the first sheet again, and so on. The three pairs of coordinates (ρ, θ), $(\rho, \theta + 2\pi)$, $(\rho, \theta + 4\pi)$, which before represented coincident points, are now regarded as representing three underlying points in separate sheets, and these points are mutually accessible only by a circuit around the origin (or winding point). The positive half of the initial line, being the threshold for passage from one sheet to another, is called the common branch line. The given function is now single-valued and continuous over the whole three-sheeted Riemann surface.

In an analogous way, the same function in cylindrical coordinates could be uniformized by taking the axis of z as a winding line, and the initial half plane as a branch membrane, giving rise to a three-fold Riemann space, which fills the whole of our space three times.

The first concrete example that I can find of a multiform potential in three dimensions is contained in a letter from Professor P. Appell to Professor F. Klein, published in the *Mathematische Annalen*, 1887, a few months before Klein delivered his lectures on potential theory (summer 1888). Appell cites a two-valued function that satisfies Laplace's three-dimensional equation, and has a certain circle for a singular line, such that the two values of the function interchange when a circuit is made around this winding line. He says nothing, however, about uniformizing the function by introducing another copy (or fold) of three-dimensional space, and using the singular (or winding line) as the boundary of a branch membrane (or door) from one fold into the other. In 1891 Dr. Pockels, in his treatise on the solutions of the "wave potential" equation for

various boundary conditions, pointed out the desirability, from both a mathematical and physical standpoint, of studying the multiform solutions of this equation, and the mode of uniformizing them on Riemann surfaces, or analogous three-dimensional spaces. He suggested that the Green functions for certain boundaries in such spaces might correspond to some unsolved physical problems. It is not unlikely that this idea may have been derived from Klein's lectures on potential theory (1888), of which Pockels made free use, but which I have not yet been able to consult directly, as they have been preserved only in manuscript.

Apparently the first application of the notion of a multiple space to a concrete physical problem was made by Professor A. Sommerfeld in articles in the *Mathematische Annalen*, 1894 and 1896, on heat conduction and diffraction, and in another article in the following year in the *Proceedings of the London Mathematical Society*, on "Multiform potentials in space." In these articles Sommerfeld shows among other things that certain boundary problems, in which the Kelvin image method fails to furnish the true Green function, on account of the existence of more than one pole within the given boundary, can be solved by the conception of the Riemann space such that each fold shall contain only one of the poles (or images). Moreover, he proves, by function theory methods, an important fundamental principle, which has been called Sommerfeld's theorem, namely, that Green's general theorem is applicable without modification to functions that have been uniformized in a Riemann space, provided the winding lines are excluded from the region of integration by means of thin tubes. From this he easily proves the uniqueness of the solution of Dirichlet's problem for any assigned boundary in the multiple space.

Sommerfeld first applies his generalized image method to find Green's function for a wedge-shaped region bounded by two infinite half planes and the infinite sphere, the angle of the planes being $n\pi/m$. Starting with the assigned pole, and reflecting successively with regard to the two planes until the series of images closes up, we find that there are n poles within the given boundary; hence we use an n-fold Riemann space having a pole in each fold, the straight edge being the winding line. The appropriate coordinate system is cylindrical, having the winding line as axis. By introducing into the elementary Green function a complex parameter, and then performing a

contour integration, using Cauchy's method of residues, a generalized function is obtained for each of the poles, and these are combined in such a way as to realize the conditions for the required Green's function in the assigned physical region. By a similar method the closely related problem of the diffraction of waves of light or electricity in passing through a wedge shaped medium has been solved by H. S. Carslaw (*Philosophical Magazine*, 1903).

The case in which $n = 2$ and $m = 1$ solves various potential and diffraction problems for the infinite half plane. In particular, the cases of waves issuing from a line source or a point source against the half-plane have been fully worked out by Sommerfeld, Carslaw and others. If the screen extended to infinity both ways, the usual method of images could easily be applied, as the image of the source would not be situated in the given region, and the action of the screen could be perfectly imitated by a combination of the source and its image, the screen being removed. As this is evidently not feasible in the case of the half-plane, we have to place the image in a second fold of a Riemann space, and find the proper Green function as already indicated. It is to be observed that the effects of the two sources have to be combined in an appropriate way according as the screen is perfectly reflecting or perfectly absorbing, so as to fulfill the proper conditions at the surface of the screen, which is then thought of as replaced by a branch membrane (or open door) between the two folds of the Riemann space, one of the folds being the ordinary physical space itself, and the other a copy coincident with it, but supposed capable of sustaining a wave motion of its own independent of the motion at underlying points in the other fold, and only influenced by the energy which enters and leaves by the open door.

In this connection it may be of interest to mention a criticism made by Professor K. Schwarzschild (*Mathematische Annalen*, 1901). Alluding to the case in which the screen is perfectly absorbing he remarks that while the second fold of the Riemann space does seem at first sight to carry away the motion so as to realize the proper conditions at the absorbing screen, yet there is nothing to prevent the latter fold from delivering some of this motion back into the first fold (or natural space) after another circuit around the winding line, thus disturbing the adjustment. Sommerfeld skillfully overcomes this objection (*Zeitschrift*, 1901, page 36) by substituting for the two-fold

space an n-fold one, and after forming the proper function, letting n become infinite; the limit so obtained must give a correct result, as it would take an infinite time for the energy to wind back into the first fold. This is equivalent to its passing out of the universe, and thus the perfectly absorbing screen is exactly imitated. The case of the perfectly reflecting screen is not open to any similar objection. Moreover, no objection can be raised against the general method on account of any supposed physical difficulty in the coexistence of different motions in the different folds, for, even if that be conceded, there remains a firm analytical ground for the method, in the generalized Green theorem and Dirichlet principle; and in any case it is not difficult to verify that the function finally obtained satisfies all the prescribed conditions in the assigned physical space.

In the article on multiform potentials already quoted, Sommerfeld next considered the more difficult case of an infinite plane out of which is cut an infinite strip bounded by two parallel lines. The suitable Riemann space has two winding lines, and the appropriate coordinate system is the bipolar system used so skillfully by Maxwell. In a later article (*Zeitschrift*, 1901, page 71) on "the bending of Roentgen rays," Sommerfeld points out that the function found in his previous article does not satisfy all the conditions, and that he has not yet found the true Green function for the region in question, although it is certain that one exists. It may be of interest to state, however, that this "problem of the slit" has been solved by an ingenious method of successive approximation by Schwarzschild in the paper already quoted (*Mathematische Annalen*, 1901). He applies Green's function for the infinite half-plane to each of the two given half-planes, in endless succession, the boundary conditions for each application being selected so as to neutralize the error left by the preceding one; this in turn disturbs the adjustment on the other half-plane, but it is proved that the successive errors diminish rapidly, and that the series is convergent.

The article contributed by Sommerfeld to the London Mathematical Society gave rise to a series of further developments along similar lines by British mathematicians and physicists. The following important papers may be mentioned:

"On multiform solutions of the differential equations of physical mathematics" (*Proceedings of the London Mathematical Society*, volume 30 (1898)) by H. S. Carslaw.

"On electric current sheets" (*Proceedings of the London Mathematical Society*, volume 31 (1899)) by J. H. Jeans.

"On the determination of Green's function by means of cylindrical and spherical harmonics" (*Proceedings of the Edinburgh Mathematical Society*, 1900) by J. Dougal.

"The Green function for a circular disk with applications to electrostatics" (*Cambridge Philosophical Transactions*, volume 18 (1900)) by E. W. Hobson.

"The use of contour integration in the problem of diffraction by a wedge of any angle" (*Philosophical Magazine*, 1903) by H. S. Carslaw.

"On many-valued Newtonian potentials" (*Proceedings of the London Mathematical Society*, 1904) by A. C. Dixon.

"On Sommerfeld's diffraction problem" (*Proceedings of the London Mathematical Society*, 1907) by Horace Lamb.

The curvilinear system used by Hobson in dealing with the circular disk is especially noteworthy. It is the peripolar system introduced by C. Neumann in treating of anchor ring problems. The three families of orthogonal surfaces which correspond to these coordinates consist (1) of a system of spherical bowls with the boundary of the disk as a common rim, and including the disk itself as a limiting member, (2) a system of anchor rings with the rim as a limiting member, and (3) a system of planes passing through the axis of the disk. The appropriate Riemann surface is two-sheeted, having the rim for the winding line and the disk for a branch membrane.

The next problem to be attacked by this method would perhaps be the diffraction of plane waves by an infinite plane screen perforated by a circular aperture; and the same coordinate system would naturally be tried. The elliptic aperture would be much more difficult. Then one might try a case of the still unsolved problem of discontinuous fluid motion in three dimensions, such as the following: A large mass of water (or other incompressible fluid) is moving with uniform velocity in one direction, and is partly obstructed by a fixed circular disk placed perpendicularly to the line of motion; to find the system of stream lines, especially those which pass around the edge and form a surface of discontinuity bounding the region of still water.

A similar problem may also be stated for a circular aperture in a fixed plane; and in both problems water may be replaced by air. It would seem that the idea of a Riemann surface (or

space) is well fitted to bridge over a discontinuity, by the introduction of an extra copy of space in which the discontinuity is filled up in some suitable way.

The success, however partial, which has attended the quest for mathematical functions that express more and more complicated physical relationships, tends to strengthen our faith in some deep-lying correspondence between the world of nature and the world of the pure intellect. It remains for mathematicians and physicists to discover new winding lines and open doors.

MATHEMATICAL APPOINTMENTS IN COLLEGES AND UNIVERSITIES.

BY PROFESSOR E. J. WILCZYNSKI.

AT the meeting of the Chicago Section held January 2, 1909, a committee was appointed for the purpose of investigating the possibility of improving the character of mathematical appointments in our colleges and universities. I submitted for the consideration of the members of the committee an adaption of the Italian system which appeared to me to be applicable to American conditions. My views were not shared by the other members of the committee, and our official report has been incorporated in the secretary's report of the April meeting of the Chicago Section.

The suggestions which I offered to the committee may be arranged under three heads.

First. Publication of vacancies. — There should be established in the BULLETIN a special department devoted to the announcement of vacancies. The administrative heads of all colleges and universities should be informed of the existence of this department and should be requested to avail themselves of it. The announcements should be as explicit as possible on the items of title, salary, grade and amount of work, probable date of appointment, address to which applications should be sent, date after which no applications will be considered, whether a specialist is desired or preferred, and if so in what special subject. If any of these items are subject to any uncertainty — if, for instance, as is often the case, an institution is not willing to announce a definite sum as salary because it would be willing to pay widely different salaries according to

the exigencies of the case— the vacancy and its general character should nevertheless be announced.

Second. The filling of major positions. — A major position shall be defined as one to which is attached a salary of $2,000 a year or more. It seems to me that the following plan might serve as a basis for discussion: If an institution desires the services of the Society in making a major mathematical appointment, it should send a request to the secretary describing as closely as possible the nature of the position to be filled and the kind of person desired. The secretary should acknowledge the request and receive assurance from the institution that it is willing to bear the expenses of the nominating committee, which expenses would in most cases be very small. Having received this assurance, the secretary should inform every member of the Council of the Society, requesting him to make out and send to the secretary before a specified date a list of six names (members or non-members of the Society) in order of preference, these to stand as his nominations for the nominating committee. Upon the specified day the secretary should tabulate the votes. The person receiving the highest number of votes should be appointed as chairman, and the two following him in the number of votes should be named as members of the nominating committee. It should be provided, however, that no two men employed in the same institution should be members of the committee. All ties, whether for chairmanship or membership, should be decided by the secretary. If two or more men from the same institution are voted into the same committee, the secretary should declare elected that one with the larger number of votes, or in case of a tie cast his own vote so as to decide for one of them, and then proceed to complete the committee from those next lower on the list.

If any person so elected to membership in a nominating committee is unable or unwilling to serve, the committee should be completed by election of the person next below him in the total number of votes. Acceptance of membership on the committee should imply renunciation on the part of the person accepting of any intention on his part of being himself a candidate for the position under consideration.

As soon as the committee has been finally constituted, the fact should be announced and the names of the members published in the BULLETIN. This should be coupled with a re-

quest to all applicants to forward their applications, a list of their publications and other documents to the chairman, and as far as possible also to the other members of the committee.

It should be the duty of the committee of three elected in this way to consider and weigh carefully the claims of all applicants. But beyond this, it should in all cases survey the entire field and select three or more men who, according to its opinion, would be the best available persons for filling the position under consideration. It should arrange these men in order of preference, giving a brief discussion of the reasons for its decision. The possibility of bracketing two or three men of apparently equal merit should not be excluded however. The report of the committee should then be forwarded to the institution concerned, for any action which it may choose to take. The list of names recommended by the committee should also be published in the BULLETIN in alphabetic order.

Most of the work of the committee could be done by correspondence. In many cases, however, it would be necessary, and it would always be desirable, for the committee to have an opportunity for oral discussion. It is not likely that an institution willing to take advantage of this plan would object to the expenses arising from this. I merely suggest, however, the possibility that a committee member living at a great distance might, with the approval of the President of the Society, appoint and instruct a substitute to take his place in the oral discussion. It would not be impossible to divide the country into regions and insist upon all members of a nominating committee being residents of one such region. But this plan would give rise to some very serious difficulties and grave objections.

Third. The filling of minor positions. — A minor position is defined as one worth less than $2,000 a year. I can see little prospect of success in any attempt on the part of the Society to assist in making such appointments, on account of the large number of such positions, and the great amount of work and expense that would be involved. The suggestion which has been made of requiring a small fee of applicants and of employing a clerk to look after the matter would defeat our main object of giving *expert* advice on such appointments. We might perhaps think of a standing committee of the Council appointed for a year, to meet upon the days assigned to meetings of the Society, whose duty it would be to make recommendations of this character, if the material in each case were

properly collected and sifted beforehand by an intelligent clerical assistant. It seems quite likely, however, that the careful filling of major positions would do away with any great necessity of reform in the methods of filling minor positions.

PICARD'S ALGEBRAIC FUNCTIONS OF TWO VARIABLES.

Théorie des Fonctions algébriques de deux Variables indépendantes. Par ÉMILE PICARD, Membre de l'Institute, Professeur à l'Université de Paris, et GEORGES SIMART, Capitaine de Frégate, Répétiteur à l'Ecole Polytechnique. Tome II. Paris, Gauthier-Villars, 1906. 8vo, v + 528 pp.

As the general scope of this great work has been explained and discussed in a very full review of volume I,* it only remains to give an idea of the large amount of additional material contained in the second volume, which is more than twice the size of the first.

Three principal objects are aimed at. The first is to present a systematic account of the theory of algebraic surfaces as developed by the Italian mathematicians. This theory deals chiefly with systems of curves on any given surface and certain numerical invariants connected with them. The other objects of the book have in view the transcendental side of the theory and deal with two of its main problems, namely, (1) the theory of the double integrals of the second kind, and (2) the integrals of total differentials of the third kind.

The work cannot be regarded as a treatise, since the subject matter with which it deals is still in a formative state and not yet sufficiently matured to admit of such a treatment. Its leading aim is rather to present in an accessible and connected form the numerous investigations of Professor Picard in this field. It accordingly consists in large part of reprints of memoirs and notes previously published in various journals together with such changes and additions as are needed to unite them into a systematic whole. A number of other notes relating to the subject, but not bearing directly on the three problems just mentioned, are collected together in an appendix of twenty pages. An additional note of about forty pages entitled "Sur

* By Mr. A. Berry in this BULLETIN, 2d series, vol. 5 (June, 1899), pp. 438–451.

quelques résultats nouveaux dans la théorie des surfaces algébriques" is contributed by Professors Castelnuovo and Enriques. Its purpose is sufficiently indicated by the title.

The work under review has the peculiar interest and fascination which attaches to a new and broad theory in process of development. The matter is presented with that clearness, precision, and elegance which characterize all the productions of the distinguished author. The reader finds the subject growing under his very eyes. This progression is strikingly evident from the fact that the present volume has appeared in three parts in 1900, 1904, and 1906, the delay in issuing the work having been caused by the need of additional investigations for the purpose of filling important gaps in the theory. So that questions are constantly springing up the solutions of which are reached in subsequent chapters of the book.

The first chapter treats of Noether's theorem relative to the curves and surfaces passing through the points of intersection of two given ones. For the case of three variables the proof here formulated is quite different from that given by Noether. One feature of this proof is, in a variety of forms, characteristic of the methods of the book as a whole. It consists in giving a constant value to one of the variables, then applying results which are known for the case of two variables.

The second chapter is a presentation of the familiar theory of point groups on an algebraic curve. This, together with Chapter III on linear systems of curves in a plane, is preparatory to a similar study of surfaces.

In Chapter IV we begin the first problem of generalization by a study of linear systems of surfaces of a given order. Such a system is defined by the behavior of the individual surfaces in certain base lines and isolated base points. The system is complete or incomplete according as all or only part of the surfaces of the given degree which behave in the prescribed manner are included.

Linear systems of surfaces which are adjoint or subadjoint to a given surface f play an especially important role in the whole work. A surface ϕ is said to be subadjoint to f if any plane section of ϕ is adjoint to the corresponding plane section of f. This definition (as well as that of adjoint), which is quite different from that given by Enriques, lends itself readily to the transcendental theory which is the main object of the book. This may readily be seen from the fact that if ϕ is a subadjoint to f, then the double integral

(1) $$\int\int \frac{\phi dx dy}{f'_z}$$

is finite at all finite points of f excepting a certain limited number of singular points of the surface.

Adjoint surfaces $Q = 0$ of order $m - 4$ are such that when the polynomial Q replaces ϕ in (1), the integral is of the first species, that is, it is everywhere finite. This is called the canonical system. Each surface of the system behaves like a subadjoint of order $m - 4$ along the base lines, and has in addition a certain behavior at the singular points of f determined by the condition that the double integral shall be finite. The number of linearly independent adjoints Q is an invariant and is denoted by p_g. Adjoint surfaces $q = 0$ of order $m - 4 + r$ are defined so that the integral $\int\int q dx dy / f'_z$ is finite at every finite point of f, and hence they behave in the multiple lines and points of f in just the same way as adjoints of order $m - 4$.

Suppose that the surfaces adjoint to f and of order h greater than $m - 4$ are cut by an arbitrary plane. The system of plane curves of order h thus obtained may not be a complete system. Let ω_h denote its default, this being the difference between the number of linear parameters of a complete system and the number of parameters in the specified system. This default becomes zero when h exceeds a certain finite integer $l - 1$. A new invariant p_n for the surface f, called the numeric genus, is introduced and is found to be connected with p_g by the relation

$$p_g - p_n = \sum_{h=m-3}^{l-1} \omega_h.$$

A remarkable theorem, due to Castelnuovo, states that when ω_{m-3} is zero all the other defaults are zero at the same time and in that case the numeric genus is equal to the geometric genus.

Chapter V takes up the linear systems of curves on a given surface $f = 0$. These are formed by the intersection of f with a linear system of surfaces. Such a system may have base points and fundamental curves. The latter occur when the general curve of the system breaks up into a variable part, and a fixed part to which this name is given. A fundamental method in the study of f is to transform it birationally into a

surface f' in hyperspace so that the curves of a given system $|C|$ on f become the curves of section of f' by hyperplanes. Of special interest are those linear systems of curves which have no base points or fundamental curves. These always exist. When such a system is used in passing from f to f' we may obtain, by successive and properly chosen projections of f' into spaces of lower dimensions, a surface F in three-dimensional space which has only ordinary singularities (double curves having triple points) and which has a (1, 1) correspondence with f.

The important place occupied by the adjoint and subadjoint surfaces becomes more manifest from the theorem that any linear system of curves can always be obtained as the intersections of f with a linear system of adjoint or subadjoint surfaces of sufficiently high order, subject, if necessary, to pass through certain bases.

Another theorem of fundamental importance is this: If an irreducible linear system of curves on f is not complete, there exists one, and but one, complete system which contains it totally. The proof given by MM. Picard and Simart is strikingly brief and elegant. It is almost intuitive in character and is a great simplification of the quite complicated proof given by Enriques. This is one of various instances in which the theory has been simplified and brought out in greater clearness in the present work.

The theory of point groups, which plays such an important part in the case of plane curves, is extended to curves on a surface f. Any curve of a given linear system is intersected by the other curves of the same system in a series of point groups called the characteristic series. The canonical series is formed by the groups of $2\pi - 2$ variable points, π being the genus of the general curve C of the system. This series is determined by means of the following theorem which holds for $p_g > 0$: An arbitrary canonical curve on f (that is, one cut out by an adjoint surface of order $m - 4$) intersects the general curve C of an irreducible system $|C|$ (one whose general curve does not break up) in a group of points which, added to a group of the characteristic series of C and to the base points of the system each counted with its degree of multiplicity, constitutes a group of the canonical series.

For each linear system $|C|$ of curves there exists a certain other linear system $|C_a|$ of particular interest called the adjoint system. A very important and beautiful property of the adjoint

system is that in the case $p_g > 0$ every irreducible linear system which admits an adjoint system is contained in this adjoint, and the residual system with respect to the adjoint is the canonical system. Accordingly each system of curves on f has, with respect to its adjoint, a residual system which is independent of the particular system chosen. Conversely, if on a given surface any irreducible system of curves is contained in its adjoint, then every other irreducible system is also contained in its adjoint.

If p_g is zero, the system $|C|$ is not contained in its adjoint system $|C_a|$. But it may happen that the system $|2C|$ (defined as that system whose base points are the same as those of $|C|$ but with double the multiplicity) is contained in $|2C_a|$. In that case the same property will be possessed by every linear system of the surface. The system which is residual to $|2C|$ with respect to $|2C_a|$ is the same, whatever system $|C|$ is chosen. This residual system is called bicanonical. The number of linearly independent bicanonical curves is an invariant P called the bi-genus. The importance of the bi-genus is due to a theorem by Castelnuovo which says that the necessary and sufficient condition for a surface to be unicursal is $p_a=0$, $P=0$.

Space will not permit us to mention other interesting results contained in this part of the work. Suffice it to say that in the 200 pages devoted to surfaces and curves on them, including the long note referred to above which completes and brings up to date the most important points in this theory, we have an admirably lucid and well considered presentation in a moderate compass of the theory of algebraic functions of two variables from the algebraic-geometric point of view.

In Chapter VII, which is a reprint with scarcely any alterations of a memoir by Picard in the *Journal de Máthematiques* for 1889, we begin with the transcendental part of the subject. The double integrals of the second kind on the algebraic surface $f(x, y, z) = 0$ are here defined to be of the form

$$\iint R(x, y, z)dx\,dy, \tag{2}$$

in which R is a rational function so constructed that when an arbitrary point A is taken on the surface, it is possible to find two rational functions U and V such that the difference between (2) and the integral

$$\iint \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\,dy \tag{3}$$

remains finite in the vicinity of A. The functions U, V may vary with the point A. It is supposed that A is not at infinity. This condition can always be satisfied by making, if necessary, a suitable transformation. If A is a multiple point of f, it is always possible to divide the region about A into a number of regions each of which corresponds birationally to a simple region on another surface. If in each of these simple regions the integral has the property just described, it will be said to have the property of an integral of the second kind in the multiple point A. In order to justify and to make (2) more clearly a generalization of the familiar abelian integral of the second kind, the author remarks that it has the property of invariance relative to birational transformations. Moreover the same form of definition can be applied to the simple abelian integral $\int R(x, y)dx$. For, such an integral is of the second kind provided it is possible to find a rational function U such that the difference

$$\int Rdx - \int \frac{dU}{dx}\,dx$$

is finite in the vicinity of a point of the given curve.

With regard to the abelian integral of the second kind, it is well known that every such integral can be expressed linearly in terms of a finite number $2p$ of such together with an additive term of the form

$$\int \frac{dU}{dx}\,dx.$$

The question naturally arises whether a corresponding theorem exists for the double integrals of the second kind. This is found to be the case; namely, every such integral is linearly expressible in terms of ρ_0 such integrals together with an additive term of the form (3). The demonstration of this theorem is attained as the result of a long series of transformations which, with some illustrations, forms the chief content of this chapter. The number ρ_0 is a new invariant for the surface f.

Another definition of the integral of the second kind, which is a more obvious generalization, characterizes it by the property that all its residues are zero relative to the curves C along which the rational function R becomes infinite. This definition is shown to be equivalent to the first.

In the case of a surface f with ordinary singularities the

double integral of the second kind can be reduced, by the subtraction of a suitable integral, to the form $\int Q dx dy / f'_z$ in which Q is a polynomial in x, y, z of limited degree.

It is important to be able to tell when two integrals of the second kind are essentially distinct, that is, when their difference is not expressible in the form (3). This evidently reduces to the problem of recognizing when a given rational function of x, y, z is expressible in the form $\partial U/\partial x + \partial V/\partial y$. On the solution of this problem depends the exact evaluation of ρ_0. The authors attempt no more at this point than to make clear the nature of the difficulties in the way of a solution, and remark that this question is intimately connected with the theory of integrals of total differentials of the third kind.

Chapter IX accordingly takes up the theory of this class of integrals, which have already been defined in volume I. We are here confronted with a very difficult part of the theory and some important problems are left unsolved. The integrals of the third kind have logarithmic discontinuities along certain algebraic curves on f, called logarithmic curves. The fundamental theorem concerning the existence of such integrals is this: On a surface f having only ordinary singularities, one can trace $\rho - 1$ particular algebraic irreducible curves $C_1, \cdots, C_{\rho-1}$ such that there does not exist any integral of the third kind having these and only these for logarithmic curves, but such that an integral does exist having for its sole logarithmic curves any ρth curve whatever together with the curves C. The exact determination of this number ρ appears to be a very difficult matter, the solution of which is obtained in the present work only for special classes of surfaces. The lack of a complete solution of this problem is especially unfortunate on account of its connection with the double integrals of the second kind. The number ρ is invariant under birational transformations only when the correspondence does not involve fundamental points or exceptional curves.

Another problem of interest connected with these integrals is whether in case the linear connection p_1 of the surface f is unity every integral of the third kind is expressible in the form

$$\sum A_k \log R_k(x, y, z) + P(x, y, z) \tag{4}$$

in which R_k and P are rational functions of x, y, z. There is ground for believing that such is the case. A considerable por-

tion of this chapter is devoted to the study of various special classes of surfaces for which the integrals of the third kind assume the form (4). The result is especially simple and elegant in the case of the Kummer surface.

The tenth chapter is addressed to the difficult task of determining the number ρ_0 of the double integrals of the second kind. As already observed above, the problem reduces to the consideration of the possibility of expressing Q/f'_z in the form

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}. \tag{5}$$

It is first shown that such an identity can be reduced to a form in which A and B in (5) are rational functions of x, y, z which become infinite only along the curves $C_1, \cdots, C_{\rho-1}$ previously associated with the integrals of total differentials of the third kind. It is then shown how, if such a system of curves C is once known, one may recognize when the identity in question can be satisfied and thus evaluate the number ρ_0.

These results are applicable for the effective determination of ρ_0 in a given case, but do not lead to a general law for this number. Some particular examples are given as illustrations. Thus, to mention but one, for the surface $z^2 = f(x)F(y)$ in which f and F are polynomials of degree $2p+1$ and $2q+1$ respectively, ρ_0 is found to be equal to $4pq$. A very curious property of ρ_0 is brought to light by some of these cases, — its value is influenced by the arithmetic nature of the coefficients in the equation of the given surface. Thus, in case of the surface $z^2 = f(x)f(y)$ in which f is a polynomial of degree 3, we have in general $\rho_0 = 3$. But when the coefficients in f satisfy the arithmetic conditions for the complex multiplication of the elliptic functions associated with irrationality $\sqrt{f(x)}$, then $\rho_0 = 2$. This peculiarity will evidently render a general formulation of laws relating to this invariant extremely difficult.

In spite of these obstacles, the task of determining a formula for ρ_0 is subsequently resumed and after a long but extremely interesting analysis we reach in Chapter XII the formula $\rho_0 = N - 4p - (m-1) + 2r - (\rho - 1)$, in which N is the class of the given surface f, m is its degree, r is the linear connection of f diminished by 1. This relation is deduced under the supposition that f has only ordinary singularities and, from the remark made above, it may fail for particular arithmetic values of the coefficients in f.

The course of investigation leading up to this formula involves a long discussion of the periods of double integrals of the first and second kinds. The properties of the linear differential equation E, introduced in volume I, is fundamental for this part of the work. This is the equation of order $2p$ whose solutions are the $2p$ periods of the abelian integral $\int Qdx/f'_z$ in which y is regarded as a parameter. Denoting by $\omega(y)$ any one of these periods, the integral $\int \omega(y)dy$ taken over a closed path in the plane of the complex variable y is a period A for the double integral $\int\int Qdx\,dy/f'_z$ provided that $\omega(y)$ returns to its original value after y has described the given path. The number of periods A is determined with the help of the differential equation E. The connection between the number ρ_0 and the number of periods then leads to the formula given above.

Chapter XIII is devoted to a determination of the numbers r_0 and r of linearly independent integrals of total differentials of the first and second kinds. These numbers are $r_0 = \delta$, $r = 2\delta$, in which δ is the default ω_{m-3}, all the other defaults $\omega_h (h > m-3)$ being supposed zero. This assumption is no restriction of generality as it can always be satisfied by a suitable transformation of the surface.

The last chapter, XIV, has for its object the illustration of the previous theory by applying it to a particular class of surfaces, namely, those whose coordinates are expressible as hyperelliptic functions of two parameters. This parametric representation lends itself with facility to the calculation of the different invariant numbers which form the basis of the present theory, and leads to simple expressions for the different types of integrals.

We remark, in conclusion, that while the present work brings to light with great clearness the difficulties which surround the extension of the theory of functions beyond the case of one variable, it affords, in the brilliant achievements which it records, the confident hope that this great field will soon become familiar territory. The splendid work of M. Picard, in the highly important results it embodies and the many indications it offers of problems pressing for solution, will undoubtedly be a powerful stimulus to activity in this line.

J. I. HUTCHINSON.

SHORTER NOTICES.

Correspondance d'Hermite et de Stieltjes, publiée par les soins de B. BAILLAUD, H. BOURGET. *Avec une Préface de* ÉMILE PICARD. Vol. I, xx + 477 pp.; vol. II, vi + 464 pp. 8vo. Paris, Gauthier-Villars, 1905.

IT is all too seldom that the mathematical public gets even a furtive view of the private life of its masters, or is allowed to penetrate into their ateliers and observe how their illustrious works are elaborated. In other fields of science, in literature and art, we are more fortunate. How great would be our loss if we knew nothing of a Darwin, a Goethe, or a Michel Angelo except their finished works. It is true that mathematicians are not altogether destitute of information regarding the inner life of their leaders. Who can read the biographies of Hamilton, Abel, or Jacobi, or the letters of Gauss to Schumacher and Bessel without being touched and edified?

The present collection of letters which passed between Hermite and Stieltjes between the years 1882 to 1894 forms a very precious accession to the mathematician's scanty library of personalia, and will be welcomed by all those who love to associate with the works of a great man the man himself.

A sketch of Hermite's life has already been given in the BULLETIN,* we will therefore devote here only a few lines to the all too brief career of Stieltjes. The son of a distinguished civil engineer, Thomas-Jean Stieltjes was born at Zwolle, Holland, December 29, 1856. He studied at the Ecole Polytechnique of Delft, where his great talents were recognized both by teachers and classmates. In spite of his exceptional abilities he did not graduate, being prevented by an instinctive dislike for competition, which seems to have pursued him all his life. In 1877 he entered as computer the Observatory of Leyden and later took part in the work of observation. But soon celestial mechanics and the higher mathematics began to absorb his attention, and interfere to such an extent with his routine duties, that he entertained the very bizarre idea of giving up his position (although living then in straitened circumstances), and coming to America to study under Sylvester, who was lecturing at the Johns Hopkins University.

* Vol. 13 (1906–7), pp. 182–190.

At this critical moment (1882) he had the good fortune to come under the beneficent influence of Hermite.* This was brought about by a letter, afterwards published in the *Comptes Rendus*, in which Stieltjes communicated to Hermite some of his results on the coefficients of the development of the perturbative function when the inclination of the orbits is considerable. Other letters followed, and soon the most friendly relations were established between the veteran Hermite and the youthful Stieltjes. His passion for mathematics now became so absorbing that Stieltjes the following year (1883) gave up his post in the observatory and devoted himself entirely to his favorite study. Numerous papers were the results of his efforts, and public recognition came in the form of the doctorate, honoris causa, from the University of Leyden, and an election to the Academy of Amsterdam.

But his hopes to an academic career received a cruel blow by his failure to obtain the vacant chair in mathematics at the University of Groningen, although the favorite candidate of the faculty. Writing to Hermite (March 13, 1884) he modestly states: "Probablement la raison aura été que n'ayant point eu l'occasion de suivre le chemin ordinaire, je n'ai point obtenu un grade à l'Université."

In 1885 Stieltjes left Holland for good, and took up his abode in Paris. He became a French citizen, and took his doctor's degree at the University of Paris the following year (June, 1886) presenting a thesis, "Sur les séries semi-convergentes." By virtue of his great talents, and supported by the powerful influence of Hermite, Stieltjes obtained at once a position as chargé de cours at the University of Toulouse, and three years later he received the professorship there. But his arduous and incessant labors began to undermine his health. In 1892 and 1893 he was obliged to pass the winters in the mild climate of Algiers, but without permanent benefit. He died the last day of the year 1894, with a brain teeming with fertile ideas and projects unfulfilled.

* * * * * * * * *

The letters number somewhat more than 400, and are altogether charming both on account of the frankness and freedom of their style, and on account of the great variety of the topics

* In 1892 he writes Hermite: "J'ai perdu mon père bien avant, en 1878; cette date est, pour moi, la fin d'une jeunesse heureuse; la période qui l'a suivie a été bien plus tourmentée, et pas sans difficultés."

discussed. A large share of the earlier letters treat of the theory of numbers, particularly those parts connected with the elliptic functions. Both Hermite and Stieltjes had a strong arithmetical bent. "Je vois aussi, Monsieur, que vous êtes un ami de l'arithmétique, et que vous partagez mon admiration pour Gauss et Eisenstein" writes Hermite in one of his first letters. Another time he writes to Stieltjes "Je confie maintenant à votre cœur arithmétique . . ." Stieltjes was much interested in Riemann's work on prime numbers and devoted considerable labor to prepare a doctor's thesis on this subject.* We therefore find an occasional discussion of Riemann's ζ function, supplemented in the appendix by four noteworthy letters to Mittag-Leffler. As is well known, Hermite first showed the transcendence of e, and a number of letters treat of this and π. In reply to Hermite's praise, Stieltjes writes with characteristic modesty "Quant à ma démonstration de la transcendence du nombre e, elle est évidemment fondée entièrement sur vos principes, et je n'ai fait que généraliser votre démonstration." Stieltjes efforts relative to π were without result and in a later letter he is "heureux d'avoir echappé au nombre π qui m'a causé un vrai cauchemar."

Although both Hermite and Stieltjes were deeply interested in the higher arithmetic, by far the larger part of their correspondence relates to analysis. Naturally the theory of continued fractions, to which Stieltjes contributed so much that is fundamental, is often treated. From a letter to Hermite we know that Stieltjes began their study as early as 1877 or 1878; while for Hermite such algorithms had formed a familiar instrument of research for two score years. We can well understand the benevolent interest that the aged Hermite took in the brilliant discoveries of his young friend in this field, and it is pleasing to hear Hermite break out in such heartfelt praise as this: "Vous êtes un merveilleux géomètre, les recherches nouvelles sur les fractions continues algébriques que vous me communiquez sont un modèle d'invention et d'élégance; ni Gauss, ni Jacobi ne m'ont jamais causé plus de plaisir."

A great deal of space is occupied in discussing Legendre's functions of the first and second kind, questions of development, asymptotic expressions, roots of $Q_n(x)$, etc.

Klein's article of 1890 on the roots of the hypergeometric function at once attracted Stieltjes' attention and he writes

* He afterward selected another subject, as stated above.

Hermite: "La démonstration de M. Klein est extrêmement intéressante, elle repose entièrement sur la conception des feuilles multiples de Riemann et des considérations géométriques. Cela prouve bien l'utilité de ce mode de représentation, mais je vous dirai que la géométrie y intervient trop, à mon gout. . . . Dès que j'ai vu l'énoncé du théorème je n'ai pu m'empêcher de penser qu'on pourrait y arriver par la méthode de Sturm en s'appuyant sur les relations entre les fonctions F contiguës, etc." Hermite replies "Mais la communication extrêmement intéressante que vous m'avez faite du beau théorème de M. Klein . . . m'arrache à la torpeur qui m'a envahi depuis plusieurs jours. Je vous chicanerai au sujet de votre prétention de le démontrer par la méthode de Sturm . . . Il me semble que cette méthode n'a jamais été appliquée et n'est applicable. . ." Whereupon Stieltjes in his next letter playfully begins to develop his method with: "Maintenant, permettez moi de défendre mon idée de démontrer le théorème de M. Klein à l'aide de la méthode de Sturm." A subject of unfailing interest especially in the later letters is the Γ function in all its astonishing and intricate relations. Mechanical quadrature and asymptotic developments are also favorite topics.

Thus we might go on describing these delightful mathematical causeries which follow one another in rapid and unaffected succession, but we hope that the above will give an idea of the rare treat these letters will afford.

JAMES PIERPONT.

Cartesian Plane Geometry. Part I.: *Analytical Conics.* By C. A. SCOTT, D.Sc. London, J. M. Dent and Company, 1907.

IF this most excellent book is really meant to be a text-book for schools, we are afraid that some one will now have to make a school for the text-book. If it is meant for the libraries of teachers and prospective teachers, it is as near perfection as one could wish; full of the best ways of doing things, of excellent examples and of inspiration for every real teacher. So, whatever faults we may see in the book are those which arise when we consider it as a function of the things it is meant to be as a text-book.

The book, rather unprepossessing in its outward appearance, contains 428 pages, a great number of illustrative examples, and 700 problems. Its thirteen chapters, which of course are

a continuous development of the theory of conics, are headed as follows: Introduction, Coordinates of point and line, Representation of point and line by equations, Loci and envelopes, Conics, Relation of straight lines and curves, Tangent at a point and polar properties, Bisected chords and diameters, Asymptotes, Properties of conics (summary), Change of axes, Systems of conics and miscellaneous examples.

The distinctive feature of the book is the use of line coordinates throughout. This, as the introduction states, makes many problems easier, and immensely increases the power of the subject. Geometrically, the line as element and line envelopes are as simple as the point and point loci, and it seems too bad that the unfortunate leaning of our ancestors toward the use of a point to pick their hieroglyphics and draw their pictures, should have led us to be almost void of visualization of how curves must look if their tangents satisfy certain conditions.

Other features of the book are the introduction of the ideal elements by means of ratios defining a point on a line; the use of the circular points and null lines with their relations to foci; the power of a point with respect to a curve; abridged notation; the reality of the imaginary elements; centers of curvature and evolutes; graphical solutions of equations and the determination of conics cutting a circle in a regular polygon. A great deal is said on the subject of bisected chords and diameters and on the properties of configurations arising from conormal points on a conic; in fact the long list of miscellaneous examples worked out in the last chapter has many of these problems which would be the despair of the ordinary undergraduate, *e. g.*, the problem of finding the centroid and orthocenter of a circumscribed triangle of a conic whose points of contact are conormal covers four pages of rather tedious algebra.

Space will permit mention here of but a few of the admirable arrangements and developments of this book.

First of all is the use of line coordinates to solve those problems for which they are best adapted. In this way the student is given another weapon with which to attack the subject; however, Professor Scott often dulls it for him, by the apologetic way in which she uses it, for she usually follows a simple demonstration with the use of line coordinates by a difficult one in point coordinates.

The condition of incidence of a point and line is derived

before anything is said about the equation of a point or line, which thus arises naturally. At the beginning of the book various types of coordinates are used, though no examples throughout the entire book are solved by the use of polar coordinates, and thus the student reader will fail to appreciate their power.

A good idea is the derivation in general of the equation of a conic without regard to axes

$$(x-p)^2+(y-q)^2=\frac{e^2}{l^2+m^2}(lx+my+n)^2$$

and hence arrival at once, by a proper choice of axes, which is seen analytically, at simple canonical forms for the different conics.

The general treatment of ideal and imaginary elements on the same basis as those ordinarily called real ones is well illustrated by the treatment of asymptotes as tangents at infinity.

Parametric equations for certain problems in conics and straight lines are used, and although in many instances these are introduced only to be got rid of, yet their importance is emphasized more than in some books. It seems strange that someone does not thoroughly develop this extremely important type of equations, in an elementary text-book.

The line equation of the general conic is introduced very cleverly, as follows: First the equations of the lines through the origin and the points of intersection P_1, P_2 of the conic with the line $\xi x+\eta y+1=0$ are found by means of the necessary condition of homogeneity and the abridged notation; then the condition that these lines through the origin and hence P_1, P_2 be coincident is imposed and thus the condition that $\xi x+\eta y+1=0$ be a tangent is found.

The idea of the graph of an equation seems to be greatly neglected; in fact the whole book gives one the impression that the mere picture of the equation has but slight importance in an analytic geometry. So we find very few examples of point graphs and practically none of line loci. As a natural sequence of this point of view Professor Scott omits the discussion of an equation, as such. This seems like a defect to us, both from the pedagogic and practical standpoint, for in few other ways does such simple analysis yield such good results.

As the book is essentially analytic, all pure geometry examples are put in the background. We cannot agree with this way

of presenting the subject to the immature student, for we believe that very few of the latter ever become interested at the beginning in geometry which is only analytic.

The subject of change of axes is rather unsatisfactorily treated. In the first place it is left almost to the end of the book, and thus the student is deprived of the use of a powerful tool for simplifying his problems and for avoiding what Professor Scott designates "algebraic gymnastics." Secondly the formulas of transformation are unrigorously derived, holding necessarily only for the special figures of the book. Finally all examples are worked by comparison with the general case *i.e.*, the general equation of the conic is transformed and its new coefficients found and these are used for the special cases. This undoubtedly gives good practical results; but the student would soon lose sight of what he was doing and the raison d'être of his formulas.

One other fault of the book appears to us to be that as a text-book it contains so many ideas and methods that it would be confusing to any student; *e. g.*, under diameters and reciprocal polars the same examples are worked in detail in three different ways. This makes the book invaluable for our libraries; but if we judge the young student aright, the study of these different methods would tend to confuse what ideas he might successfully gather from the most simple and hence best method of attacking the problem.

E. GORDON BILL.

Grundlagen der Geometrie. Von DAVID HILBERT. Dritte Auflage. Leipzig, Teubner, 1909. vi + 279 pp.

THE present edition of Hilbert's celebrated treatise on geometry appears as a seventh volume in a series of essays, "Wissenschaft und Hypothese," whose first volume is a translation of Poincaré's well known book bearing the latter title. It differs but little from the second edition. Two reprints of papers by Hilbert, "Ueber den Zahlbegriff" * and "Ueber die Grundlagen der Logik und Arithmetik," † appear as appendices VI and VII respectively; also additional references to investigations of other authors are inserted.

Since very full consideration has been given Hilbert's book in the BULLETIN ‡ and elsewhere § the reviewer must refrain

* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8 (1900).

† Verhandlungen des III Internationalen Kongresses in Heidelberg, 1904.

‡ See vol. 6, p. 287; vol. 9, p. 158; vol. 10, p. 1.

§ Cf. E. B. Wilson, *Archiv der Mathematik u. Physik* (3), vol 6 (1904), p. 104.

from any lengthy discussion. Some of the logical phases of Hilbert's axioms are instructively pointed out in a controversy between Frege and Korselt in the *Jahresbericht der Deutschen Mathematiker-Vereinigung.**

A. R. SCHWEITZER.

Vorträge über den mathematischen Unterricht an den höheren Schulen. KLEIN-SCHIMMACK. Teil I. *Von der Organisation des mathematischen Unterrichts.* Leipzig, 1907, pp. ix + 236.

PROFESSOR KLEIN has for years been calling attention to the fact that it is the duty of the universities to furnish instruction not only in the subject matter of mathematics but also in the questions relative to the teaching of the subject, since teaching is the profession to which the great majority of the students are looking forward; and in his own university (Göttingen) he has admirably been putting into practice what he preaches. The present volume makes accessible to the wider public the substance of a course of lectures thus delivered at Göttingen in 1904–5.

It is a work that at once commanded thoughtful attention in all quarters of the nation for which it was especially intended, but it also deserves and will receive careful study far beyond the national confines.

The work is divided into the following sections, whose titles give a general idea of its plan and scope:†

Introduction, pages 1–9; I. Elementary schools, 10–18; II. The six lower classes‡ of the (boys') higher schools, 19–43; III. Girls' schools and trade schools, 44–66; IV. The historical development of instruction in mathematics in the German higher schools, 67–99; V. The three upper classes§ of the higher schools according to the curricula of 1901, 100–126; VI. Propositions for reform in the upper classes of the higher schools, 127–159; VII. The universities and technical schools, 158–189; Conclusion, 189–190; Appendix (containing re-

* See vol. 12 (1903), pp. 319, 368, 402; vol. 15 (1906), pp. 293, 377, 423; vol. 17 (1908), p. 98.

† With each numbered title, except No. IV, the words "mathematics in" are of course implied.

‡ In mathematics these six classes correspond roughly to our grades 4 to 8, and the first year of the high school.

§ In mathematics these classes correspond roughly to the second and the third year of the high school and the freshman year in college.

prints of two of Klein's earlier papers and of the curriculum proposed by the Commission at Meran *), 191–236.

The form of the work is, as already stated, that of lectures to students of a German university preparing to teach mathematics in Germany. It therefore naturally contains much matter that relates exclusively to German schools and much of a more general character that is expressed in the terminology of the German school system; but there is also not a little, and that of the most important and characteristic, which is intelligible without any acquaintance with the details of the German organization.† The chapters of widest general interest are Nos. II, V, and VI.

As to general methods, Klein finds that the genetic method has been steadily gaining ground and has entirely displaced the unduly formalistic systems of earlier days. In this connection, the question is raised, pertinent also elsewhere than in Germany, whether "many a university lecture course might not take a lesson in pedagogic-psychologic preparation from the modern types of instruction in the schools" (page 24).

Quite a little space is devoted to a discussion of the function concept, which, in geometric form, is to be made the *soul* of the mathematical courses in the schools (page 34). In close connection with this lies the question of the interrelations in time and subject matter of the various branches of mathematics taught in the schools, a question of prime importance for us Americans, since in this matter we are far behind the Germans, who have long been teaching algebra (including numerical work) and geometry (with trigonometry) side by side, details being left in the main to the discretion of the teacher. Let us hear whether Klein is satisfied with this, or whether perchance he even believes that the Germans have gone too far in this direction.

"According to my feeling, instruction in mathematics is at present still conducted with too much external separation of algebra and geometry, and with this external separation a

*See BULLETIN, vol. 12 (1906), pp. 317–352.

† Various descriptions of the German secondary schools, quite sufficient to make the present work intelligible throughout, are within reach of the American reader. For example:

Russell: German Higher Schools. New York (Longmans). 1898.

Bolton: The Secondary School System of Germany, New York (Appleton), 1900.

Young: The Teaching of Mathematics in Prussia. New York (Longmans), 1900.

deeper one often goes hand in hand. The principle of purity of methods may be interesting in the higher special courses of the university, but unfortunately it is also sometimes proclaimed in the schools. 'Geometria geometrice'; 'No numbers in geometry'; 'Happily, algebra does not need geometric crutches'; 'By all means, no diagrams in arithmetic'; such are the mottoes for the sacrifice of thorough treatment to the higher honor of purity of method. It is needless to discuss the unpedagogic character of this position. The secondary schools, whose ideal aim is general culture, should evidently present mathematics as an organism whose various parts stand in active and vital interrelations.

"If, gentlemen, any mathematical question presents itself to you in later life you will have no leisure to ask: 'Shall I proceed purely geometrically, perhaps even projectively, avoiding all metric concepts, or shall I follow the method of the pure theory of numbers?' No; the pressing thing will be to be able to help yourself out on the basis of the totality of your mathematical acquisitions. And this will in general be easier the closer together the different forms of mathematical thought have been brought in your instruction. Just on that account it will be well if the arithmetic, algebra, and geometry of the lowest classes coalesce in 'Obertertia' and 'Untersecunda' (ages 13–15) and culminate in a grasp of the function concept. Of course, geometry is not to be sacrificed to algebra or vice versa; by mutual assistance, both forms of mathematical thinking will not lose but gain."

The fifth chapter is devoted to a discussion of the course in mathematics in the three upper classes, the last of which takes up approximately our freshman work in college. The list of subjects prescribed by the curricula for these years is taken up in detail (Algebra, pages 101–109, Geometry, pages 121–126), and the considerations are interesting and suggestive to Americans also. A large portion of the chapter (pages 109–121) is devoted to an "Excursus on the question of the infinitesimal calculus in the schools." So far as the *last* school year (our freshman year) is concerned, the reasons so cogently stated would find ready acceptance by some in America. The writer has for over a decade given an introductory course in calculus for freshmen, and no doubt the ideas of the calculus have been more or less extensively introduced into freshmen work in numerous instances. Mention should be made especially of the

work in mathematics of the first year in the Massachusetts Institute of Technology, as set forth in the excellent and suggestive work of Woods and Bailey,* which aims to organize college algebra, plane analytic geometry, and the elements of the calculus into a whole, whose unifying note shall be *mathematics* rather than any one of the subjects named. But to carry the calculus back into precollegiate work, even in the most elementary way, would not at present be practicable with us. Still Klein's proposal to introduce the ideas of derivative and integral in what corresponds roughly to our third high school year may well be feasible in Germany (where the pupils remain in the same school, and probably with the same teacher during the years in question), and has there met with strong, though by no means unanimous, support.

Mere mention must suffice for the interesting discussion of proposed reforms in Chapter VI, and the sketch (Chapter VII) of mathematics at the universities, especially Göttingen, with its graphic representation in colors of the galaxy of famous men that in unbroken sequence have justly constituted Göttingen's pride.

The work is written in an informal, clear, and interesting style, and should find a considerable circle of readers on this side of the Atlantic.

Upon the heels of the work discussed above there has followed a lithographed volume of a course of lectures by Klein (in 1907–8 †) treating topics related to elementary mathematics from the higher standpoint, along the lines of the synoptic course recommended by the German Commission in 1907.‡ This is a further valuable and stimulating fruit of Klein's attitude mentioned at the outset, and merits attention and study by a wide circle of readers. A second volume dealing with geometric topics is to follow quite shortly, and a full review of both will appear in the BULLETIN in due course.

J. W. A. YOUNG.

* A Course in Mathematics, vol. I., Boston (Ginn), 1907.

† Elementar-Mathematik vom höheren Standpunkte aus. Teil I: Arithmetik, Algebra, Analysis. Leipzig (Teubner), 1908, pp. 590.

‡ See BULLETIN, vol. 15 (1909), p. 262.

CORRECTION.

The following errata in the present volume of the BULLETIN have come to the attention of the editors:

Page 162, lines 8–9 relate to Dr. Irwin's paper, No. 6 on the preceding pages.

Page 324, lines 2–3, read "Note on Laplace's equation," read October 25, 1902.

NOTES.

BEGINNING July 1, 1909, the publication of the *Bulletin* and the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, heretofore in charge of The Macmillan Company, will be taken over by the Society. Orders and subscriptions of non-members should be sent direct to the Society at 501 West 116th Street, New York, N. Y.

A FEW copies of the reprint of the Chicago Symposium on mathematics for engineering students are still on hand and may be procured from the Secretary of the Chicago Section, Professor H. E. Slaught, University of Chicago.

AT the meeting of the London mathematical society held on May 13, the following papers were read: By H. W. TURNBULL, "Ternary quadratic types"; by J. G. LEATHEM, "The theorem of Gauss in the theory of attraction"; by J. E. LITTLEWOOD, "Criteria for continuity and discontinuity of a function defined by an infinite product."

The committee on a syllabus of geometry for secondary schools recently appointed by the Council of the American federation of teachers of the mathematical and natural sciences in connection with the mathematics section of the department of secondary education of the National education association has been arranged as follows: Professors H. E. Slaught, chairman, C. L. Bouton, F. Cajori, H. E. Hawkes, E. R. Hedrick, H. L. Rietz, D. E. Smith, Messrs. Wm. Betz, E. L. Brown, W. B. Carpenter, W. W. Hart, F. K. Newton, E. R. Smith, R. L. Short, and Miss Mabel Sykes.

ON the occasion of the second centenary of the birth of Leonhard Euler, in 1907, the Swiss society of naturalists ap-

pointed a committee to investigate and report on the feasibility of publishing all his memoirs. The fourth international congress of mathematicians, in Rome, 1908, passed a resolution commending the action of the Swiss society, and urging academies, mathematical societies, and individual mathematicians to assist in the undertaking. On August 30, 1908, the committee reported its readiness to undertake the task, provided sufficient financial assistance could be secured. Before making a public request, the committee first addressed itself to the Swiss people, from whom it has received about 100,000 francs. At the Cologne meeting of the Deutsche Mathematiker-Vereinigung the sum of 5,000 francs was appropriated for the same purpose. The Paris academy of sciences has subscribed for forty complete sets. It is estimated that the total cost of publication will be 400,000 francs; the complete works will comprise about forty volumes, and are offered at subscription at not to exceed 25 francs per volume, to be paid upon appearance of the single volumes.

With this much already accomplished, the society now addresses itself to the world, with the special request that academies, mathematical societies, physical societies, associations of engineers and of industrial undertakings follow, so far as they can, the example of the organizations mentioned. In particular, libraries of institutions of learning, of insurance companies, and of cities are invited to send in their subscriptions at once. Moreover, voluntary contributions from individuals will be of great assistance to the committee. All subscriptions and contributions should be sent to Professor Ferdinand Rudio, Zürich V, Dolderstrasse 111.

THE eighty-first convention of German naturalists and physicians will be held at Salzburg, Austria, September 19–25, 1909. The Deutsche Mathematiker-Vereinigung will hold its annual summer meeting in affiliation with it, under the presidency of Professor M. KRAUSE, of the technical school at Dresden. Titles and abstracts of papers to be read at this meeting should be sent to the secretary, Professor A. KRAZER, Westendstrasse 57, Karlsruhe, as soon as possible. Reports on algebraic functions of several variables and the theory of integral equations will be presented.

WITH the new editorial organization of the *American Mathematical Monthly* the University of Illinois joins the University

of Chicago in assisting in its financial support. The *Monthly* does not desire to compete with other mathematical periodicals, but hopes to offer an opportunity for less mature writers to express views on points of interest and to publish original solutions, etc., which are of value, yet hardly appropriate for more ambitious journals.

THE following courses in advanced mathematics are announced for the academic year 1909–1910:

UNIVERSITY OF GÖTTINGEN (first semester). — By Professor F. KLEIN: Projective and non-euclidean geometry, four hours; Seminar, two hours. — By Professor D. HILBERT: Partial differential equations, four hours; Seminar, two hours. — By Professor C. RUNGE: Mechanics, four hours; Seminar, two hours. — By Professor L. PRANDTL: Thermodynamics, two hours; Seminar, two hours. — By Professor E. LANDAU: Theory of functions, four hours; Theory of prime ideals, one hour. — By Professor W. VOIGT: Electrodynamics, four hours. — By Professor E. ZERMELO: Logical foundations of mathematics, two hours. — By Professor F. BERNSTEIN: Calculus of probabilities, two hours. — Fourier's series, two hours. — By Dr. O. TOEPLITZ: Determinants, with applications to analytic geometry of space, four hours. — By Dr. P. KOEBE: Algebra, four hours; Differential equations, two hours.

INDIANA UNIVERSITY. — By Professor S. C. DAVISSON: Theory of surfaces, three hours; Differential equations, three hours (a, w). — By Professor D. A. ROTHROCK: Theory of functions, three hours (a, w); Advanced calculus, three hours; History of mathematics, three hours (s). — By Professor U. S. HANNA: Invariants, two hours. — By Professor C. HASEMAN: Applications of partial differential equations, three hours. (a = autumn, w = winter, s = spring.)

INDIANA UNIVERSITY (summer quarter, June 24 to September 3, 1909). — By Professor D. A. ROTHROCK: Theory of equations, five hours; Advanced calculus, five hours. — By Professor U. S. HANNA: Theory of numbers, five hours. — By Professor C. HASEMAN: Vector analysis, five hours.

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Solid analytic geometry, two hours; Higher plane curves, three hours; Mathematics of insurance, two hours. — By Professor G. E. FISHER: Advanced calculus, two hours;

Calculus of variations, two hours. — By Professor I. J. SCHWATT: Infinite series and products, two hours; Definite integrals, three hours. — By Professor G. H. HALLETT: Modern higher algebra, three hours (first half year); Galois theory of equations, three hours (second half year); Theory of groups of a finite order, three hours; Lie's theory of continuous groups, three hours (first half year). — By Professor F. H. SAFFORD: Mathematical theory of precision of measurements, three hours (first half year); Curvilinear coordinates, three hours (second half year). — By Dr. O. E. GLENN: Invariants and covariants, three hours.

THE Belgian academy of sciences announces the following prize subjects for 1910:

A systematic and didactic treatise on recent advances in the theory of partial differential equations of the second order is desired. Prize, 800 francs.

Analyze and complete the investigations made in the calculus of variations since 1850. Prize 600 francs.

Manuscripts should be written in French or in Flemish, and submitted to the secretary of the academy under the usual conditions before August 1, 1910.

THE Jablonowski society of Leipzig announces the following prize problem for the year 1911:

"It is desired that the theory of the rainbow should be extended, especially the distribution of light over the surface of a sphere, with a stated degree of approximation. The diameter of the sphere should be small enough to avoid bending of the spectrum bands, and large enough to be at least comparable with the length of light waves."

PROFESSOR G. LORIA, of the University of Genoa, has been elected foreign member of the Bohemian academy of sciences of Prague.

PROFESSOR T. LEVI-CIVITA, of the University of Padua, and Professor C. ARZELÀ, of the University of Bologna, have been elected corresponding members of the mathematical society at Charkow.

SINCE the complete destruction of the University of Messina by the earthquake, various professors have been temporarily assigned to duties in technical schools, academic institutes, and other schools. Professor T. BOGGIO has been appointed professor of mechanics at the institute of Florence.

PROFESSOR C. CARATHÉODORY, of the University of Bonn, has accepted a professorship of mathematics at the technical school at Hanover.

PROFESSOR J. HADAMARD, of the University of Paris, has been appointed professor of analytic mechanics at the Collège de France, as successor to the late M. Lévy.

PROFESSOR M. KUTTA, of the technical school at Munich, has accepted a full professorship of applied mathematics at the University of Jena.

MR. J. F. CAMERON has been appointed tutor in Gonville and Caius College, Cambridge.

PROFESSOR W. L. TANNER, of the University of Monmouthshire, South Wales, has resigned after holding the chair of mathematics for twenty-six years.

PROFESSOR G. H. LING, of Columbia University, has been appointed head of the department of mathematics in the newly established University of Saskatchewan.

AT the University of Minnesota, Dr. A. L. UNDERHILL, of the University of Wisconsin, and Professor G. P. PAINE, of Ripon College, have been appointed assistant professors of mathematics. Mr. J. S. MIKESH, of the University of Colorado, has been appointed instructor in mathematics. Mr. R. R. SHUMWAY has been granted a year's leave of absence for further study. Professor G. A. BAUER will return from Europe this summer to resume his position.

AT Yale University Dr. W. R. LONGLEY has been promoted to an assistant professorship of mathematics. Dr. L. I. HEWES has resigned his position as instructor in mathematics.

THE following changes have taken place at Princeton University: Professor J. H. JEANS has resigned to accept a professorship of mathematical physics at the University of Cambridge; Professor L. P. EISENHART has been promoted to a full professorship of mathematics; Professor C. E. STROMQUIST has resigned, having accepted the professorship of mathematics at the University of Wyoming.

PROFESSOR F. S. WOODS, of the Massachusetts Institute of Technology, has been granted leave of absence for the academic year to study in Europe. Dr. A. J. LENNES returns to the Institute as instructor in mathematics.

AT the University of Colorado Professor S. EPSTEEN has been appointed Secretary of the Graduate School. Professor Epsteen has also been elected president of the Colorado mathematical society.

THE following instructors in mathematics have been appointed at the University of Wisconsin: Dr. A. DRESDEN, in the department of letters and science: Dr. H. E. BUCHANAN, Mr. E. E. MOOTS, and Mr. E. H. MOULTON in the school of engineering.

PROFESSOR C. S. SLICHTER, of the University of Wisconsin, has been granted leave of absence during the academic year 1909–1910 to study in Europe.

PROFESSOR O. BIERMANN, of the German technical school at Brünn, died April 28, at the age of 51 years.

PROFESSOR P. MUTH, the author of the Elementarteiler, died April 30 at Osthofen, Rheinhessen, at the age of 48 years.

PROFESSOR G. VAILATI died at Rome, May 15, at the age of 46 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (W. R.). Récréations mathématiques et problèmes des temps anciens et modernes. Vol. III. Paris, Hermann, 1909. 8vo. Fr. 5.00

BOREL (E.). Eléments de la théorie des probabilités. Paris, Hermann, 1909. 8vo. 6 + 200 pp. Fr. 6.00

BURGGRAF (G.). Die Γ-Function für komplexe Argumente. Brünn, 1909. 8vo. 33 pp.

D'ADHÉMAR (R.). L'équation de Fredholm. Problèmes de Dirichlet et de Neumann. Paris, Hermann, 1909. 8vo. Fr. 3.50

DZIOBEK (O.). Lehrbuch der analytischen Geometrie. Teil I: Analytische Geometrie der Ebene. 2te verbesserte Auflage. Braunschweig, Graff, 1909. 8vo. 8 + 358 pp. M. 4.50

HASS (P.). Zur Definition des Begriffs der eindeutigen analytischen Funktion. (Diss.) Kiel, 1908. 8vo. 55 pp.

JORDAN (C.). Cours d'analyse de l'école polytechnique. 3e édition, revue et corrigée. Vol. I. Paris, Gauthier-Villars, 1909. 8vo. 16 + 621 pp. Fr. 17.00

KAMBLY und ROEDER. Elemente der analytischen Geometrie der Ebene. Neu bearbeitet von A. Thaer. Breslau, Hirt, 1909. 8vo. 121 pp. M. 1.30

KOJALOWITSCH (B.). Lectures upon higher mathematics. (Russian.) St. Petersburg, 1909. 8vo. 376 pp. M. 9.00

MARGOSSIAN (A.). De l'ordonnance des nombres dans les carrés magiques impairs. Paris, Hermann, 1909. 8vo. 140 pp. Fr. 2.00

MELLOR (J. W.). Higher mathematics for chemistry and physics. London, 1909. 8vo. 664 pp. Cloth.

MITTAG-LEFFLER (F.). Niels-Henrik Abel. Paris, Hermann, 1909. 8vo. 68 pp. Fr. 2.00

PICARD (E.). Traité d'analyse. Vol. 3. 2e édition. Paris, Gauthier-Villars, 1908. 8vo. 7 + 607 pp. Fr. 18.00

PFITZNER (P.). Ueber den didaktischen Wert zusammenhängender Aufgabengruppen im mathematischen Unterricht. Dresden, 1909. 4to. 32 pp. M. 1.20

SCHÖNER (E.). Studien zur Wurflinie. Erlangen, 1909. 8vo. 67 pp. M. 2.40

TANNERY (J.). Elemente der Mathematik. Deutsch von P. Klaess. Leipzig, Teubner, 1909. 8vo. 12 + 339 pp. M. 7.00

VEGAS (M.). Elementos de geometria analitica. Barcelona, Gili, 1909. 8vo. 240 pp. P. 4.00

WALEK (K.). Binäre kubische Transformation und Complexe. (Diss.) München, 1908. 8vo. 39 pp.

WINKLER (J.). Die Schwingungsfiguren in analytischer Behandlung. Celle, 1909. 8vo. 32 pp. M. 1.50

II. ELEMENTARY MATHEMATICS.

ECKHARDT (E.). Zurückführung der sphärischen Trigonometrie auf die Geometrie des ebenen Kreisvierecks. Neue Grundlegung für die Formeln der sphärischen Trigonometrie. Leipzig, Teubner, 1909. 8vo. 6 + 155 pp. M. 4.40

GEOMETRY and graphic algebra. London, Wyman, 1909. 1 d.

GIROD (F.) et WEILL (E.). Eléments de géométrie à l'usage des élèves du premier cycle des lycées et collèges et des classes de première et seconde A et B conformes au programme du 27 juillet, 1905. Paris, André, 1909. 8vo. 399 pp.

KAMBLY und ROEDER. Stereometrie. Neu bearbeitet von A. Thaer. 32ste Auflage, Breslau, Hirt, 1909. 8vo. 167 pp. Cloth. M. 2.00

PITMAN'S guide to intermediate and advanced scholarship examinations with answers to the mathematical sections for students preparing for scholarship examinations. London, Pitman, 1909. 12mo. 140 pp. 2s.

ROMERO RUBIERA (A.). Elementos de trigonometria. Alicante, López, 1909. 8vo. 77 pp. P. 1.00

THIEME (H.). Die Elemente der Geometrie. Leipzig, Teubner, 1909. 8vo. 12 + 394 pp. Cloth. M. 9.00

III. APPLIED MATHEMATICS.

BOULANGER (A.). Hydraulique générale. Vol. II: Problèmes à singularités et applications. Paris, Doin, 1909. 12mo. 7 + 299 pp.

CONTALDI (P.). La meccanica nella scuola e nell' industria. Vol. II. 2a edizione, riveduta ed ampliata. Milano, Hoepli, 1909. 8vo. 15 + 745 pp. L. 16.00

COSSERAT (E.) et (F.). Théorie des corps déformables. Paris, Hermann, 1909. 8vo. 6 + 228 pp. Fr. 6.00

MATISSE (G.). Le principe de la conservation de l'assise et ses applications. Paris, Hermann, 1909. 8vo. 65 pp. Fr. 2.50

NELTING (R.). Der nautisch-astronomische und Universal-Rechenstab zur mechanischen Lösung aller mathematischen Probleme, unter Berücksichtigung der terrestrischen und astronomischen Navigation. Hamburg, Friederichsen, 1909. 8vo. 6 + 67 pp. M. 4.00

RICHARZ (F.). Anfangsgründe der Maxwellschen Theorie verknüpft mit der Elektronentheorie. Leipzig, Teubner, 1909. 8vo. 9 + 246 pp. M. 7.00

WEISSE (E.). Anwendung der elliptischen Funktionen auf ein Problem aus der Theorie der Gelenkmechanismen. Rostock, 1909. 8vo. 102 pp. M. 2.50

EIGHTEENTH ANNUAL LIST OF PAPERS.

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

AMES, L. D. A Simpler Proof of Lie's Theorem for Ordinary Differential Equations. Read (Chicago) Apr. 9, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 8, pp. 384–386; May, 1909.

BILL, E. G. The Construction of a Space Field of Extremals. Read Dec. 30, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 8, pp. 374–378; May, 1909.

BIRKHOFF, G. D. Boundary Value and Expansion Problems of Ordinary Linear Differential Equations. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 4, pp. 373–395; Oct., 1908.

—— Existence and Oscillation Theorem for a Certain Boundary Value Problem. Read (Chicago) Jan. 1, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 259–270; Apr., 1909.

BLISS, G. A. See MASON, M.

BÔCHER, M. On the Regions of Convergence of Power Series Which Represent Two-Dimensional Harmonic Functions. Read Oct. 25, 1902. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 271–278; Apr., 1909.

BOLZA, O. The Determination of the Conjugate Points for Discontinuous Solutions in the Calculus of Variations. Read (Chicago) Mar. 30, 1907. *American Journal of Mathematics*, vol. 30, No. 3, pp. 209–221; July, 1908.

—— Heinrich Maschke: His Life and Work. Read Sept. 10, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 2, pp. 85–95; Nov., 1908.

BUCHANAN, H. E. On Certain Determinants Connected with a Problem in Celestial Mechanics. Read (Chicago) Jan. 2, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 5, pp. 227–231; Feb., 1909.

CARMICHAEL, R. D. A Table of the Values of m Corresponding to Given Values of $\phi(m)$. Read (Chicago) Mar. 30, 1907. *American Journal of Mathematics*, vol. 30, No. 4, pp. 394–400; Oct., 1908.

—— Even Multiply Perfect Numbers of Five Different Prime Factors. Read Apr. 27, 1907. *Bulletin of the American Mathematical Society*, vol. 15, No. 1, pp. 7–8; Oct., 1908.

—— On the Geometric Properties of Quartic Curves Possessing Fourfold Symmetry with Respect to a Point. Read Oct. 26, 1907. *Annals of Mathematics*, Ser. 2, vol. 10, No. 2, pp. 81–87; Jan., 1909.

—— On the General Tangent to Plane Curves. Read (Chicago) Apr. 18, 1908. *American Mathematical Monthly*, vol. 15, Nos. 8-9, pp. 153–157; Aug.–Sept., 1908.

—— On Plane Algebraic Curves Symmetrical with Respect to Each of Two Rectangular Axes. Read (Chicago) Apr. 18, 1908. *American Mathematical Monthly*, vol. 15, No. 10, pp. 175–177; Oct., 1908.

—— Notes on the Simplex Theory of Numbers. Read Oct. 31, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 5, pp. 217–223; Feb., 1909.

—— On the Theory of Functions of a Triple Variable. Read Oct. 31, 1908. *American Mathematical Monthly*, vol. 16, No. 3, pp. 41–46; Mar., 1909.

CARVER, W. B. The Quadric Spreads Connected with the Configuration $\Gamma^{n+2}_{n+4,n}$, and a Special Case in the Pascal Hexagram. Read Dec. 28, 1906, and Sept. 5, 1907. *American Journal of Mathematics*, vol. 31, No. 1, pp. 1–17; Jan., 1909.

—— Degenerate Pencils of Quadrics Connected with $\Gamma^{n+2}_{n+4,n}$ Configurations. Read Dec. 31, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 10, pp. 483–486; July, 1909.

COBLE, A. B. An Application of the Form Problems Associated with Certain Cremona Groups to the Solution of Equations of Higher Degree. Read Dec. 27, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 4, pp. 396–424; Oct., 1908.

—— Symmetric Binary Forms and Involutions. Read Dec. 31, 1908. *American Journal of Mathematics*, vol. 31, No. 2, pp. 183–212; Apr., 1909.

DICKSON, L. E. Invariantive Reduction of Quadratic Forms in the $GF[2^n]$. Read (Chicago) Dec. 28, 1906. *American Journal of Mathematics*, vol. 30, No. 3, pp. 263–281; July, 1908.

—— On Commutative Linear Groups. Read Sept. 6, 1907, and (Chicago) Jan. 1, 1908. *Quarterly Journal of Pure and Applied Mathematics*, vol. 40, No. 2–3, pp. 167–196; Jan.–May, 1909.

—— On Families of Quadratic Forms in a General Field. Read (Chicago) Dec. 30, 1907. *Quarterly Journal of Pure and Applied Mathematics*, vol. 39, No. 4, pp. 316–333; July, 1908.

—— Definite Forms in a Finite Field. Read Sept. 10, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 1, pp. 109–122; Jan., 1909.

—— Rational Reduction of a Pair of Binary Quadratic Forms; their Modular Invariants. Read Sept. 10, 1908. *American Journal of Mathematics*, vol. 31, No. 2, pp. 103–146; Apr., 1909.

—— On the Last Theorem of Fermat. Read Sept. 11, 1908. *Messenger of Mathematics*, vol. 38, Nos. 1–2, pp. 14–32; May–June, 1908, and *Quarterly Journal of Pure and Applied Mathematics*, vol. 40, No. 1, pp. 27–45; Oct., 1908.

—— General Theory of Modular Invariants. Read (Chicago) Jan. 1, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 123–158; Apr., 1909.

—— On the Representation of Numbers by Modular Forms. Read (Chicago) Jan. 2, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 7, pp. 338–347; Apr., 1909.

DINES, L. L. A Method of Investigating Numbers of the Forms $6^k.s \pm 1$. Read (Chicago) Apr. 17, 1908. *Annals of Mathematics*, ser. 2, vol. 10, No. 3, pp. 105–115; Apr., 1909.

DRESDEN, A. The Second Derivatives of the Extremal Integral. Read Sept. 10, 1908. *Transactions of the American Mathematical Society*, vol. 9, No. 4, pp. 467–486; Oct., 1908.

FORD, W. B. A Set of Criteria for the Summability of Divergent Series. Read (Chicago) Apr. 9, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 9, pp. 439–444; June, 1909.

GRANVILLE, W. A. Duality in the Formulas of Spherical Trigonometry. Read Dec. 31, 1908. *American Mathematical Monthly*, vol. 16, No. 3, pp. 47–53; Mar., 1909.

GRIFFIN, F. L. On the Law of Gravitation in the Binary Systems. Read Dec. 28, 1906, and Sept. 6, 1907. *American Journal of Mathematics*, vol. 31, No. 1, pp. 62–85; Jan., 1909.

—— Families of Central Orbits Related to Circular Trajectories. Read Sept. 10, 1908. *American Mathematical Monthly*, vol. 16, No. 4, pp. 57–66; Apr., 1909.

HAIGTWELL, G. W. Plane Fields of Force Whose Trajectories are Invariant under a Projective Group. Read (Southwestern Section) Nov. 28, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 220–246; Apr., 1909.

HASEMAN, C. Anwendung der Theorie der Integralgleichungen auf einige Randwertaufgaben der Funktionentheorie. Read (Chicago) Dec. 30, 1907. *Mathematische Annalen*, vol. 66, No. 2, pp. 258–272; Nov., 1908.

HASKINS, C. N. On the Second Theorem of the Mean. Read Feb. 29, 1908. *Annals of Mathematics*, ser. 2, vol. 9, No. 4, pp. 173–179; July, 1908.

HEDRICK, E. R. A Smooth Closed Curve Composed of Rectilinear Segments with Vertex Points which are Nowhere Dense. Read Sept. 5, 1907. *Annals of Mathematics*, ser. 2, vol. 9, No. 4, pp. 163–166; July, 1908.

HODGE, F. H. See MOULTON, E. J.

HOSKINS, L. M. General Algebraic Solutions in the Logic of Classes. Read (San Francisco) Feb. 29, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 2, pp. 82–83; Nov., 1908.

—— A General Diagrammatic Method of Representing Propositions and Inference in the Logic of Classes. Read (San Francisco) Feb. 29, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 2, pp. 84–85; Nov., 1908.

HUTCHINSON, J. I. The Hypergeometric Functions of N Variables. Read Apr. 25, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 1, pp. 61–70; Jan., 1909.

IRWIN, F. The Invariants of Linear Differential Expressions. Read Oct. 31, 1908. *Proceedings of the American Academy of Arts and Sciences*, vol. 44, No. 1, pp. 1–60; Nov., 1908.

KASNER, E. Natural Families of Trajectories: Conservative Fields of Force. Read Apr. 28, 1906, and Apr. 27, 1907. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 201–219; Apr., 1909.

—— Tautochrones and Brachistochrones. Read Feb. 27, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 10, pp. 475–483; July, 1909.

LANDRY, A. E. A Geometrical Application of Binary Syzygies. Read Oct. 31, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 1, pp. 95–108; Jan., 1909.

LOVETT, E. O. On a Group of Transformations which Occurs in the Problem of Several Bodies. Read Dec. 29, 1902, and Sept. 5, 1907. *American Journal of Mathematics*, vol. 30, No. 4, pp. 307–324; Oct., 1908.

LUNN, A. C. The Foundations of Trigonometry. Read (Chicago) Dec. 29, 1905. *Annals of Mathematics*, ser. 2, vol. 10, No. 1, pp. 37–45; Oct., 1908.

——Some Notes on Vector Analysis. Read (Chicago) Jan. 2, 1909. *American Mathematical Monthly*, vol. 16, No. 2, pp. 21–25; Feb., 1909.

MCKINNEY, T. E. Concerning a Certain Type of Continued Fractions Depending on a Variable Parameter. Read Aug. 19, 1901. *American Journal of Mathematics*, vol. 29, No. 3, pp. 213–278; July, 1907.

MANNING, W. A. On the Order of Primitive Groups. Read (San Francisco) Sept. 29, 1906. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 247–258; Apr., 1909.

MASON, M. On the Linear Differential Equation of Hyperbolic Type. Read Sept. 3, 1906. *Mathematische Annalen*, vol. 65, No. 4, pp. 570–575; June, 1908.

MASON, M., and BLISS, G. A. The Properties of Curves in Space Which Minimize a Definite Integral. Read Sept. 16, 1904, and Oct. 27, 1906. *Transactions of the American Mathematical Society*, vol. 9, No. 4, pp. 440–466; Oct., 1908.

MILLER, G. A. Methods to Determine the Primitive Roots of a Number. Read Sept. 5, 1907. *American Journal of Mathematics*, vol. 31, No. 1; pp. 42–44; Jan., 1909.

—— On the Multiple Holomorphs of a Group. Read (Chicago) Dec. 30, 1907. *Mathematische Annalen*, vol. 66, No. 1, pp. 133–142; Aug., 1908.

—— Generalization of Positive and Negative Numbers. Read (San Francisco) Feb. 29, 1908. *American Mathematical Monthly*, vol. 15, Nos. 6–7, pp. 117–119; June-July, 1908.

—— The Central of a Group. Read (Chicago) Apr. 18, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 1, pp. 50-60; Jan., 1909.

—— Answer to a Question Raised by Cayley as Regards a Property of Abstract Groups. Read Sept. 10, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 2. pp. 72–75; Nov., 1908.

—— On the Groups Generated by Two Operators s_1, s_2 Satisfying the Equation $s_1 s_2 = s_2^\alpha s_1^\beta$. Read Oct. 31, 1908. *Quarterly Journal of Pure and Applied Mathematics*, vol. 40, No. 3, pp. 197–209; May, 1909.

—— On the Groups Generated by Two Operators Satisfying the Condition $s_1 s_2 = s_2^{-\gamma} s_1^{-\delta}$. Read (Southwestern Section) Nov. 28, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 4, pp. 162–165; Jan., 1909.

—— Finite Groups Which May be Defined by Two Operators Satisfying Two Conditions. Read Dec. 30, 1908. *American Journal of Mathematics*, vol. 31, No. 2, pp. 167–182; Apr., 1909.

MILLER, K. G. See WHITE, H. S.

MOORE, R. L. Sets of Metrical Hypotheses for Geometry. Read (Chicago) Apr. 22, 1905. *Transactions of the American Mathematical Society*, vol. 9, No. 4, pp. 487–512; Oct., 1908.

MOREHEAD, J. C. Extension of the Sieve of Eratosthenes to Arithmetical Progressions and Applications. Read Apr. 29, 1905, and (Chicago) Dec. 29, 1905. *Annals of Mathematics*, ser. 2, vol. 10, No. 2, pp. 88–104; Jan., 1909.

MOULTON, E. J., and HODGE, F. H. On Certain Properties of the Orbits of a Particle Subject to a Central Force Varying as an Integral Power of the Distance. Read (Chicago) Apr. 18, 1908. *American Mathematical Monthly*, vol. 15, Nos. 6–7, pp. 119–130; June-July, 1908.

MOULTON, F. R. On Certain Relations among the Possible Changes in the Motions of Mutually Attracting Spheres when Distributed by Tidal Interactions. Read (Chicago) Jan. 1, 1908. *Publications of the Carnegie Institution of Washington*, No. 107; Jan., 1909.

NEWSON, H. B. Transformazione projettive ad un Parametro e loro Gruppi continui. Read (Chicago) Dec. 29, 1905. *Giornale di Matematiche di Battaglini*, vol. 43, pp. 33–62; 1905.

RANUM, A. On Periodic Linear Substitutions Whose Coefficientsare Integers. Read Sept. 6, 1907. *Bulletin of the American Mathematical Society*, vol. 15, No. 1, pp. 4–6; Oct., 1908.

—— The Group Membership of Singular Matrices. Read Dec. 27, 1907. *American Journal of Mathematics*, vol. 31, No. 1, pp. 18–41; Jan., 1909.

RICHARDSON, R. G. D. The Integration of a Sequence of Functions and its Application to Iterated Integrals. Read Apr. 28, and Sept. 3, 1906, and Apr. 27, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 3, pp. 339–372; July, 1908.

ROEVER, W. H. Brilliant Points of Curves and Surfaces. Read Feb. 29, 1908. *Transactions of the American Mathematical Society*, vol. 9, No. 3, pp. 245–279; July, 1908.

SAUREL, P. On Fredholm's Equation. Read Apr. 24, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 9, pp. 445–450; June, 1909.

SCHUR, I. Beiträge zur Theorie der Gruppen linearer homogener Substitutionen. Read (Chicago) Apr. 18, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 159–175; Apr., 1909.

SCHWEITZER, A. R. An Interesting Class of Monotonic Functions. Read (Chicago) Dec. 28, 1906. *American Mathematical Monthly*, vol. 16, No. 1, pp. 4–9; Jan., 1909.

—— Note on Enriques's Review of the Foundations of Geometry. Read (Chicago) Apr. 18, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 6, pp. 314–315; Mar., 1909.

SHAW, J. B. Standard Forms of Certain Types of Peirce Algebras. Read (Chicago) Dec. 30, 1907. *American Journal of Mathematics*, vol. 31, No. 1, pp. 45–61; Jan., 1909.

SINCLAIR, M. E. The Absolute Minimum in the Problem of the Surface of Revolution of Minimum Area. Read Sept. 6, 1907. *Annals of Mathematics*, ser. 2, vol. 9, No. 4, pp. 151–155; July, 1908.

SNYDER, V. On the Range of the Birational Transformations of Curves of Genus Greater Than the Canonical Form. Read Sept. 6, 1907. *American Journal of Mathematics*, vol. 30, No. 4. pp. 337–346; Oct., 1908.

—— Normal Curves of Genus Six, and Their Groups of Birational Transformations. Read Feb. 29, 1908 *American Journal of Mathematics*, vol. 30, No. 4, pp. 325–336; Oct., 1908

—— Construction of Plane Curves of Given Order and Genus, Having Distinct Double Points. Read Sept. 10, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 1, pp. 1–4; Oct., 1908.

—— Surfaces Derived from the Cubic Variety Having Nine Double Points in Four-Dimensional Space. Read Sept. 11, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 1, pp. 71–78; Jan., 1909.

—— Surfaces and Congruences Derived from the Cubic Variety Having a Double Line in Four-Dimensional Space. Read Dec. 31, 1908. *American Journal of Mathematics*, vol. 31, No. 2, pp. 147–166; Apr., 1909.

STEPHENS, R. P. On the Pentastroid. Read Feb. 23 and Sept. 6, 1907. *American Journal of Mathematics*, vol. 30, No. 4, pp. 381–393; Oct., 1908.

STUDY, E. Zur Differentialgeometrie der analytischen Curven. Read Sept. 10, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 1, pp. 1–49; Jan., 1909.

UNDERHILL, A. L. Invariants of the Function $F(x, y, x', y')$ in the Calculus of Variations. Read Apr. 27, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 3, pp. 316–338; July, 1908.

—— The Second Variation of a Definite Integral. Read (Chicago) Jan. 1, 1908, and Jan. 2, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 8, pp. 379–384; May, 1909.

VEBLEN, O. Continuous Increasing Functions of Finite and Transfinite Ordinals. Read Sept. 6, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 3, pp. 280–292; July, 1908.

VEBLEN, O., and YOUNG, J. W. A Set of Assumptions for Projective Geometry. Read Dec. 27, 1907. *American Journal of Mathematics*, vol. 30, No. 4, pp. 347–380; Oct., 1908.

WESTFALL, W. D. A. Note on the Theorem of Generalized Fourier's Constants. Read (Southwestern Section) Dec. 1, 1906. *Bulletin of the American Mathematical Society*, vol. 15, No. 2, pp. 76–79; Nov., 1908.

WESTLUND, J. On the Equation $x^n + y^n = nz^n$. Read (Chicago) Apr. 17, 1908. *American Mathematical Monthly*, vol. 16, No. 1, pp. 3–4; Jan., 1909.

WHITE, H. S. Bézout's Theory of Resultants and its Influence on Geometry. Read Dec. 30, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 7, pp. 325–338; Apr., 1909.

WHITE, H. S., and MILLER, K. G. Note on Lüroth's Type of Plane Quartic Curves. Read Sept. 6, 1907. *Bulletin of the American Mathematical Society*, vol. 15, No. 7, pp. 347–352; Apr., 1909.

WILCZYNSKI, E. J. Projective Differential Geometry of Curved Surfaces (Third Memoir). Read (Chicago) Jan. 1, 1908. *Transactions of the American Mathematical Society*, vol. 9, No. 3, pp. 293–315; July, 1908.

—— Projective Differential Geometry of Curved Surfaces (Fourth Memoir). Read Apr. 18, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 176–200; Apr., 1909.

WILSON, E. B. On the Theory of Double Products and Strains in Hyperspace. Read Sept. 4, 1906, and Dec. 28, 1907. *Transactions of the Connecticut Academy of Arts and Sciences*, vol. 14, No. 1, pp. 1–57; Sept., 1908.

—— On the Differential Equations of the Equilibrium of an Inextensible String. Read Apr. 25, 1908. *Transactions of the American Mathematical Society*, vol. 9, No. 4, pp. 425–439; Oct., 1908.

—— On the Principle of Relativity. Read Apr. 25, 1908. *Philosophical Magazine*, ser. 6, vol. 16, No. 93, pp. 419–422; Sept., 1908.

—— Note on Statistical Mechanics. Read Sept. 11, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 3, pp. 107–115; Dec., 1908.

—— The Number of Types of Collineations. Read Oct. 31, 1908. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 17, Nos. 9–10, pp. 341–344; Sept.–Oct., 1908.

YOUNG, A. E. On a Certain Class of Isothermic Surfaces. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 10, No. 1, pp. 79–94; Jan., 1909.

YOUNG, J. W. See VEBLEN, O.

INDEX.

AMES, L. D. A Simpler Proof of Lie's Theorem for Ordinary Differential Equations, 384.

BILL, E. G. The Construction of a Space Field of Extremals, 374.

—— See REVIEWS, under Scott.

BLISS, G. A. See REVIEWS, under Lebon.

BOLZA, O. Heinrich Maschke: His Life and Work, 85.

—— Remarks Concerning the Second Variation for Isoperimetric Problems, 213.

BROWN, E. W. See REVIEWS, under Annuaire, Jeans.

BUCHANAN, H. E. On Certain Determinants Connected with a Problem in Celestial Mechanics, 227.

BUSSEY, W. H. See REVIEWS, under Arnoux.

CAJORI, F. See REVIEWS, under Voss.

CARMICHAEL, R. D. Even Multiply Perfect Numbers of Five Different Prime Factors, 7.

—— Notes on the Simplex Theory of Numbers, 217.

CARVER, W. B. Degenerate Pencils of Quadrics Connected with $\Gamma^{n+2}_{n+4,n}$ Configurations, 483.

COLE, F. N. Reports of Meetings of the American Mathematical Society: October Meeting, 157; Fifteenth Annual Meeting, 275; February Meeting, 323; April Meeting, 419.

COWLEY, E. B. See REVIEWS, under Laurent.

CRATHORNE, A. R. See REVIEWS, under Scheffers.

DICKSON, L. E. On the Representation of Numbers by Modular Forms, 338.

—— See REVIEWS, under Minkowski, Sylvester, Tannery.

DODD, E. L. See REVIEWS, under Bois.

DOWLING, L. W. See REVIEWS, under Czuber.

FORD, W. B. A Set of Criteria for the Summability of Divergent Series, 439.

HASKINS, C. N. See REVIEWS, under Osgood.

HAWKES, H. E. See REVIEWS, under Enriques.

HEWES, L. I. See REVIEWS, under d'Ocagne.

HOSKINS, L. M. General Algebraic Solutions in the Logic of Classes, 82.

—— A General Diagrammatic Method of Representing Propositions and Inference in the Logic of Classes, 84.

HUTCHINSON, J. I. See REVIEWS, under Durège, Picard.

KASNER, E. Tautochrones and Brachistochrones, 475.

—— See REVIEWS, under Loria, Schubert.

KELLOGG, O. D. The Second Regular Meeting of the Southwestern Section, 207.

LONGLEY, W. R. See REVIEWS, under Andoyer, Jouguet.

LYTLE, E. B. See REVIEWS, under Slaught.

MCMAHON, J. On the Use of n-Fold Riemann Spaces in Applied Mathematics, 486.

MANNING, W. A. Reports of Meetings of the San Francisco Section of the American Mathematical Society: September Meeting, 105; February Meeting, 371.

MILLER, G. A. Answer to a Question Raised by Cayley as Regards a Property of Abstract Groups, 72.

—— On the Groups Generated by Two Operators Satisfying the Condition $s_1 s_2 = s_2^{-2} s_1^{-2}$, 162.

—— The Sixtieth Meeting of the American Association for the Advancement of Science, 303.

—— See REVIEWS, under Müller, Royal Society.

MILLER, K. G. See WHITE, H. S.

MOORE, C. L. E. The Fourth International Congress of Mathematicians; Sectional Meetings. 8.

—— See REVIEWS, under Fabry.

MOORE, C. N. On Certain Constants Analogous to Fourier's Constants, 116.

MOULTON, F. R. See REVIEWS, under Poincaré.

OSGOOD, W. F. See REVIEWS, under Goursat.

OWENS, F. W. See REVIEWS, under Whitehead.

PIERPONT, J. See REVIEWS, under Hedrick, Hermite, Veblen.

RANUM, A. On Periodic Linear Substitutions Whose Coefficients are Integers, 4.

—— See REVIEWS, under Hilton.

RICHARDSON, R. G. D. The Cologne Meeting of the Deutsche Mathematiker-Vereinigung, 117.

RIETZ, H. L. See REVIEWS, under Blaschke.

SAUREL, P. On Fredholm's Equation, 445.

SCHWEITZER, A. R. On the Logical Basis of Grassmann's Extensive Algebra, 79.

—— Note on Enriques's Review of the Foundations of Geometry, 314.

—— See REVIEWS, under Hilbert.

SLAUGHT, H. E. The Fifteenth Summer Meeting of the American Mathematical Society, 57.

—— Reports of Meetings of the Chicago Section of the American Mathematical Society: Winter Meeting, 293; April Meeting, 425.

SMITH, D. E. See REVIEWS, under Ball, Cantor, Gray, Günther, Heath, White.

SMITH, E. M. Some Surfaces Having a Family of Helices as One Set of Lines of Curvature, 309.

SNYDER, V. Construction of Plane Curves of Given Order and Genus, Having Distinct Double Points, 1.

—— See REVIEWS, under Scheibner, Schoenflies, Sturm.

TYLER, H. W. See REVIEWS, under Chicago Symposium.

UNDERHILL, A. L. The Second Variation of a Definite Integral, 379.

WESTFALL, W. D. A. Note on the Theorem of Generalized Fourier's Constants, 76.

—— The Solution of Boundary Problems of Linear Differential Equations of Odd Order, 223.

—— A Class of Functions Having a Peculiar Discontinuity, 225.

WHITE, H. S. Bézout's Theory of Resultants and its Influence on Geometry, 325.

WHITE, H. S., AND MILLER, K. G. Note on Lüroth's Type of Plane Quartic Curves, 347.

WILCZYNSKI, E. J. Mathematical Appointments in Colleges and Universities, 492.

—— See REVIEWS, under Hesse.

WILSON, E. B. Note on Statistical Mechanics, 107.

—— See REVIEWS, under Fisher, Fleming, Grimsehl.

WRIGHT, J. E. See REVIEWS, under Carslaw.

YOUNG, J. W. See REVIEWS, under Bachmann.

YOUNG, J. W. A. See REVIEWS, under Gutzmer, Klein.

REVIEWS.

Andoyer, H. Cours d'Astronomie. Seconde Partie: Astronomie pratique, W. R. LONGLEY, 467.

Annuaire du Bureau des Longitudes pour l'An 1909, E. W. BROWN, 364.

Arnoux, G. Arithmétique graphique. Les Espaces arithmétiques, leurs Transformations, W. H. BUSSEY, 255.

Aschkinass, E. See Fleming, J. A.

Bachmann, P. Grundlehren der neueren Zahlentheorie, J. W. YOUNG, 463.

Baillaud, B. See Hermite, C.

Baker, H. F. See Sylvester, J. J.

Ball, W. W. R. Histoire des Mathématiques (traduite par L. Freund), Tome 2, D. E. SMITH, 244.

Blaschke, E. Vorlesungen über mathematische Statistik (Die Lehre von den statistischen Masszahlen), H. L. RIETZ, 142.

Bois, G. P. Tafeln unbestimmter Integrale, E. L. DODD, 364.

Bourget, H. See Hermite, C.

Cantor, M. Vorlesungen über Geschichte der Mathematik, Band 4, D. E. SMITH, 352.

Carslaw, H. S. Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat, J. E. WRIGHT, 196.

Chicago Symposium on Mathematics for Engineering Students, H. W. TYLER, 450.

Czuber, E. Vorlesungen über Differential- und Integral-Rechnung, Band 2 (2nd edition), L. W. DOWLING, 392.

Durège, H. Theorie der elliptischen Funktionen (fünfte Auflage, von L. Maurer), J. I. HUTCHINSON, 132.

Enriques, F. Fragen der Elementargeometrie (Deutsche Ausgabe von H. Fleischer), Theil 2, H. E. HAWKES, 256.

Fabry, E. Traité de Mathématiques générales a l'Usage des Chimistes, Physiciens, Ingenieurs, et des Elèves des Facultés des Sciences, C. L. E. MOORE, 395.

Fisher, I. The Nature of Capital and Income. The Rate of Interest. E. B. WILSON, 169.

Fleischer, H. See Enriques, F.

Fleming, J. A. Elektrische Wellen-Telegraphie (Deutsche Ausgabe von E. Aschkinass), E. B. WILSON, 143.

Freund, L. See Ball, W. W. R.

Goursat, E. Cours d'Analyse mathématique, volume 2, W. F. OSGOOD, 120.

Gray, G. J. A Bibliography of the Works of Sir Isaac Newton together with a List of Books Illustrating his Works, D. E. SMITH, 188.

Grimsehl, E. Angewandte Potentialtheorie in elementarer Behandlung, E. B. WILSON, 198.

Gundelfinger, S. See Hesse, O.

Günther, S. Geschichte der Mathematik, Theil 1, D. E. SMITH, 245.

Gutzmer, A. Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Aerzte, J. W. A. YOUNG, 261.

Heath, T. L. The Thirteen Books of Euclid's Elements translated from the Text of Heiberg with Introduction and Commentary, D. F. SMITH, 386.

Hedrick, E. R. An Algebra for Secondary Schools, J. PIERPONT, 134.

Hermite, C. Correspondance d'Hermite et de Stieltjes, publieé par B. Baillaud et H. Bourget, volumes 1–2, J. PIERPONT, 504.

Hesse, O. Vorlesungen aus der analytischen Geometrie der geraden Linie, des Punktes, und des Kreises in der Ebene (Vierte Auflage, von S. Gundelfinger), E. J. WILCZYNSKI, 186.

Hilbert, D. Grundlagen der Geometrie (Dritte Auflage), A. R. SCHWEITZER, 510.

Hilton, H. An Introduction to the Theory of Groups of Finite Order, A. RANUM, 239.

Jeans, J. H. An Elementary Treatise on Theoretical Mechanics, E. W. BROWN, 165.

Jouguet, E. Lectures de Mécanique, Partie 1, W. R. LONGLEY, 466.

Klein, F., and Schimmack, R. Vorträge über den mathematischen Unterricht an den höheren Schulen, Theil 1, J. W. A. YOUNG, 511.

Laurent, H. La Géométrie analytique générale, E. B. COWLEY, 363.

Lebon, E. Table de Caractéristiques relatives a la Base 2310 des Facteurs premiers d'un Nombre inférieure à 30,030, G. A. Bliss, 139.

Lennes, N. J. See Veblen, O., and Slaught, H. E.

Loria, G. Il Passato ed il Presente delli principali Teorie geometriche (Terza Edizione), E. Kasner, 402.

Maurer, L, See Durège, H.

Minkowski, H. Diophantische Approximationen. Eine Einführung in die Zahlentheorie, L. E. Dickson, 251.

Müller, F. Führer durch die mathematische Literatur mit besonderer Berücksichtigung der historisch wichtigen Schriften, G. A. Miller, 403.

Ocagne, M. d'. Calcul graphique et Nomographie. Le Calcul simplifié. L. I. Hewes, 127.

Osgood, W. F. A First Course in the Differential and Integral Calculus, C. N. Haskins, 457.

Picard, E., and Simart, G. Théorie des Fonctions algébriques de deux Variables indépendantes, J. I. Hutchinson, 495.

Poincaré, H. Leçons de Mécanique céleste professées a la Sorbonne, F. R. Moulton, 258.

Royal Society of London. Catalogue of Scientific Papers, 1800–1900. Subject Index, volume 1, Pure Mathematics, G. A. Miller, 192.

Scheffers, G. Serrets Lehrbuch der Differential- und Integralrechung (Dritte Auflage), A. R. Crathorne, 140.

Scheibner, W. Beiträge zur Theorie der linearen Transformationen als Einleitung in die algebraische Invariantentheorie, V. Snyder, 137.

Schoenflies, A. Einführung in die Hauptgesetze der zeichnerischen Darstellungsmethoden, V. Snyder, 362.

Schimmack, R. See Klein, F.

Schubert, H. Auslese aus meiner Unterrichts- und Vorlesungspraxis, E. Kasner, 400.

Scott, C. A. Cartesian Plane Geometry, Part 1: Analytical Conics, E. G. Bill, 507.

Serret, J. A. See Scheffers, G.

Simart, G. See Picard, E.

Slaught, H. E., and Lennes, N. J. High School Algebra, E. B. Lytle, 357.

Sturm, R. Die Lehre von den geometrischen Verwandtschaften, V. Snyder, Erster Band, 135; Zweiter Band, 252.

Sylvester, J. J. Collected Mathematical Papers (edited by H. F. Baker), Volumes 1–2, L. E. Dickson, 232.

Tannery, J. Manuscrits de Evariste Galois, L. E. Dickson, 249.

Veblen, O., and Lennes, N. J. Introduction to Infinitesimal Analysis. Functions of One Real Variable, J. Pierpont, 133.

Voss, A. Ueber das Wesen der Mathematik, F. Cajori, 405.

White, W. F. A Scrapbook of Elementary Mathematics, D. E. Smith, 190.

Whitehead, A. N. The Axioms of Descriptive Geometry, F. W. Owens, 465.

Correction, 145, 515.

Index, 530.

New Publications, 51, 101, 151, 202, 269, 319, 368, 416, 471, 520.

Notes, 43, 95, 145, 198, 264, 316, 365, 410, 468, 515.

Papers read before the Society and Subsequently Published, Eighteenth Annual List of, 523.

NOTES AND OTHER ITEMS.

Academies, Associations, Congresses, and Societies:

American Mathematical Society: Annual Meeting, 95, 145; Annual Register, 198; Chicago Section, 145, 264, 365; Chicago Symposium, 264, 515; Colloquium, 198, 468; Election of Officers, 276; New Members Admitted, 57, 157, 276, 323, 419; Presidential Address, 95, 145; Publishers, 515; Southwestern Section, 95; Summer Meeting, 198, 468; Transactions, 43, 96, 264, 323, 410, 419, 515.

Associations for the Advancement of Science: American, 265; Australasian, 147; British, 96, 468; French, 96; Italian, 146; Spanish, 317.

Associations of Teachers of Mathematics: American Federation, 145, 469, 515; International Commission, 265, 316, 411; Middle States, Syracuse Section, 265; Swiss, 199.

Astronomische Gesellschaft, 318; Belgian Academy, 44, 518; Berlin Academy, 46; Circolo Matematico di Palermo, 265, 365; Deutsche Mathematiker-Vereinigung, 316, 365, 516; Göttingen Academy, 44, 45; International Congress of Philosophy, 199; Italian Scientific Society, 98; Jablonowski Society, 518; Lombardy Royal Institute, 45; London Mathematical Society, 146, 199, 316, 410, 468; Mathesis, 44; National Academy, 470; Paris Academy, 265; Turin Academy, 317.

Books, Announcements of New, 46, 200, 264, 267, 411, 515.

Catalogues of Books, Models, etc., 46, 100, 147, 150, 268, 318, 319, 411, 471.

Doctorates in Mathematics: American, 97; Paris, 318.

Euler's Works, Publication of, 515.

Gauss Memorial, 318.

Journals:

American Journal of Mathematics, 43, 96, 316, 410; American Mathematical Monthly, 411, 516; Annals of Mathematics, 43, 145, 265, 410; Transactions of the American Mathematical Society, 43, 96, 264, 323, 410, 419, 515.

Papers and Communications Presented to the Society, Authors:

Allardice, R. E., 371, 371.
Ames, L. D., 58, 427.
Baker, R. P., 426.
Bates, W. H., 295, 427.
Bill, E. G., 277, 278.
Blichfeldt, H. F., 105.
Bliss, G. A., 58, 278, 294.
Bolza, O., 58.
Brenke, W. C., 207.
Buchanan, H. E., 295, 295, 426.
Buck, T., 426.
Carmichael, R. D., 157, 157, 277, 277, 419, 419.
Carver, W. B., 278.
Chessin, A. S., 420.
Coble, A. B., 278.
Conner, J. R., 278.
Craig, C. F., 278.
Davis, E. W., 207, 426.
Dederick, L. S., 420.
Denton, W. W., 295.
Dickson, L. E., 58, 59, 59, 294, 295, 426, 427, 427.
Dowling, L. W., 59.
Dresden, A., 58.
Eiesland, J., 277.

Eisenhart, L. P., 420.
Fite, W. B., 59, 278.
Ford, W. B., 426, 426.
Frizell, A. B., 207, 208, 294, 295.
Gaba, M., 207.
Gillespie, D. C., 323.
Glenn, O. E., 58.
Glover, J. W., 294.
Granville, W. A., 278.
Griffin, F. L., 58, 277.
Hardy, J. G., 278.
Hartwell, G. W., 208.
Haskins, C. N., 59, 278, 426.
Hedrick, E. R., 59, 208.
Hewes, L. I., 59.
Hildebrandt, T. H., 427.
Hutchinson, J. I., 277.
Ingold, L., 58.
Irwin, F., 158.
Jackson, W. H., 420, 420.
Kasner, E., 59, 157, 278, 278, 323, 420.
Keyser, C. J., 278.
Landry, A. E., 158.
Leib, D. D., 278.
Lipke, J., 277.
Longley, W. R., 59, 277.
Lovett, E. O., 277.
Lunn, A. C., 294, 295, 295.
McDonald, J. H., 105.
McEwen, G. F., 371.
Maclagan-Wedderburn, J. H., 59.
MacMillan, W. D., 427, 427.
Manning, W. A., 371.
Mason, M., 58.
Miles, E. J., 294, 294.
Miller, G. A., 58, 59, 157, 207, 277, 426.
Moore, C. L. E., 277.
Moore, C. N., 59.
Moore, E. H., 295.
Morehead, J. C., 420, 426, 426.
Moreno, H. C., 105.
Morley, F., 277.
Moulton, F. R., 427, 427.
Neikirk, L. I., 427.
Newson, H. B., 59.
Noble, C. A., 105.
Osgood, W. F., 420.
Pell, A. J., 427.
Phillips, H. B., 278.
Pitcher, A., 427.
Ranum, A., 58, 58, 105, 278, 278, 426.
Rietz, H. L., 427.
Roe, E. D., 420.
Roever, W. H., 207.
Saurel, P. L., 420.
Schweitzer, A. R., 295.
Sharpe, F. R., 58, 277.
Shaw, J. B., 58, 295.
Sinclair, M. E., 27 .
Snyder, V., 58, 58, 278, 323.
Stager, H. W., 371.
Study, E., 58.
Townsend, E. J., 59.
Underhill, A. L., 295.
Van der Vries, J. N., 207.
Wernicke, A. L. P., 207, 208.
Western, A. E., 426.
White, H. S., 277.
Whitney, A. W., 371.
Wilczynski, E. J., 59, 294, 427.
Wilson, E. B., 59, 158.
Wright, J. E., 420.
Young, A. E., 426.
Young, J. W., 59, 295.

Personal Notes:

Almansi, E., 98; Appell, P., 96; Archibald, R. C., 150, 201; Arzelà, C., 518.

Bagnera, G., 367; Bauer, G. A., 519; Bianchi, L., 266; Biermann, O., 520; Biklé, C. E., 51; Bill, E. G., 97; Birkhoff, G. D., 319; Bliss, G. A., 100, 198, 468; Bôcher, M., 470, 471; Boggio, T., 49, 518; Bolza, O., 470; Bonola, R., 98; Börger, R. L., 97; Bourgeois, M., 99; Boutroux, P., 201; Brown, E. W., 265; Browne, J. J., 416; Buchanan, H. E., 520; Buhl, E., 367; Bullard, J. J., 100; Burgatti, P., 49, 415; Burkhardt, H., 49, 149; Butterfield, A. D., 100.

Cameron, J. F. 519; Carathéodory, C., 150, 518; Carrus, P., 367; Chambers, G. G., 97; Chase, F. L., 266; Chessin, A. S., 99; Cisotti, U., 201; Claussen, H., 46; Clements, G. R., 150; Cleveland, J. B., 100; Cobb, C. W., 100; Conant, L. L., 100; Conwell, G. M., 51, 97; Cowley, E. B., 97; Craig, C. F., 97; Crathorne, A. R., 150.

Dalaker, H. H., 50; Darboux, G., 200; Darwin, F., 96; Darwin, G. H., 200; Davisson, S. C., 50; Debierne, —, 266; DeFranchis, G., 49; Delassus, M., 319; Dickson, L. E., 411; Dirichlet, P. G. L., 318;

Disteli, M., 268, 367 ; Dresden, A., 520 ; Dulac, M., 49, 99 ; Duval, E. P. R., 50.

Eisenhart, L. P., 519 ; Elkin, W. L., 266 ; Enache, N., 318 ; Epsteen, S., 519 ; Epstein, P., 49 ; Esclangon, —, 150.

Fabry, E., 266 ; Fehr, H., 199 ; Fenssner, W., 99 ; Field, F., 51 ; Fields, J. C., 468 ; Finkel, B. F., 412 ; Fréchet, M., 48 ; Fredholm, I., 266 ; Fricke, R., 99, 149 ; Frizell, A. B., 51 ; Fubini, G., 367.

Gaba, M., 99 ; Geckeler, O. T., 51 ; Gerbaldi, F., 49 ; Glaisher. J. W. L., 50 ; Goursat, M., 201 ; Griffin, F. L., 470 ; Greenhill, A. G., 50 ; Guichard, C., 266 ; Gutzmer, A., 49.

Hadamard, J., 266, 519 ; Hanawalt, F. W., 51 ; Hartwell, G. W., 99 ; Haseman, C., 50 ; Hellinger, M., 201 ; Henderson, A., 99 ; Henry, G. K. G., 99 ; Hensel, K., 49, 149 ; Herglotz, G., 49, 415 ; Hertzer, H., 268 ; Hessenberg, G., 201 ; Heywood, H. B., 318 ; Hewes, L. I., 519 ; Hill, G. W., 200 ; Hines, J. C., 99 ; Hodge, F. H., 99 ; Holder, F. J., 97 ; Howland, L. A., 150 ; Huntington, E. V., 264 ; Husson, H., 99.

Ingold, L., 97 ; Irwin, F., 50, 97.

Jeans, J. H., 198, 468, 519 ; Johnson, A. R., 150 ; Joly, H., 51 ; Jordan, D. S., 265.

Kabitz, W., 46 ; Kasner, E., 198, 468 ; Kenyon, A. M., 150 ; Keppel, H. G., 50 ; Kienast, —, 415 ; Kneser, A., 49 ; Kollros, L., 268 ; Korkin, A., 150 ; Krygowsky, —, 367 ; Kutta, W., 470, 519.

Lacombe, M., 99 ; Laisant, C. A., 201 ; Lalescu, T., 318 ; Lamb, H., 50 ; Lancrenon, P. M. E., 266 ; Landau, E., 367 ; Lebert, E., 266 ; Lebesgue, H., 201 ; Lebon, E., 97 ; Legoux, E. A., 319 ; Leib, D. D., 367 ; Lemoine, E., 266 ; Lennes, N. J., 50, 97, 201, 519 ; Levi, E. E., 415 ; Levi-Civita, T., 518 ; Lévy, M., 367 ; Liapounoff, A., 268 ; Ling, G. H., 415, 519 ; Lipke, J., 50 ; Longley, W. R., 519 ; Lorentz, H. A., 99 ; Loria, G., 200, 415, 518 ; Luck, J. J., 98 ; Ludwig, W., 415 ; Luizet, M., 266 ; Lytle, E. B., 98, 100.

McKelvey, J. V., 416 ; McKinney, T. E., 100 ; Maclaurin, R. C., 150 ; MacNeish, H. F., 100 ; Maillet, E., 201 ; Mann, C. R., 145, 469 ; Meissner, —, 415 ; Mikesh, J. S. 519 ; Miller, E. A., 50 ; Miller, G. A., 265, 412 ; Miller, J. W., 50 ; Minkowski, H., 319 ; Mises, R. v., 367 ; Moore, C. L. E., 50, 470 ; Moore, C. N., 98 ; Moore. E. H., 265 ; Moore, R. L., 100 ; Moots, E. E., 520 ; Moreno, H. C., 415 ; Morera, G., 367 ; Moulton, E. H., 520 ; Müller, C , 201 ; Muth, P., 520.

Neumann, E., 149 ; Nordmann, C., 266.

Osgood, W. F., 265, 470 ; Owens, F. W., 98.

Paine, G. P., 519 ; Pasquier, — du, 415 ; Petot, —, 266 ; Petr, K., 48 ; Phillips, E. C. F., 98 ; Picard, E., 48 ; Picciati, G., 98 ; Pierce, C. A., 99 ; Pieri, M., 49 ; Pierson, I. C., 51 ; Plemelj, J., 367 ; Poincaré, H., 99, 267, 367 ; Popovici, C., 318 ; Porter, M. B., 150 ; Prohazka, F., 48 ; Puiseux, P., 266.

Rau, R., 318 ; Redfield, J. H., 100 ; Reye, T., 49, 367 ; Richardson, R. G. D., 150 ; Riggs, N. C., 100 ; Ritter, P., 46 ; Roever, W. H., 50 ; Rothrock, D. A., 50 ; Rousier, J. B., 318 ; Rupp, O., 319 ; Rusl, F., 318 ; Rutherford, E., 468.

Scarborough, J. H., 98 ; Schilling, F., 415 ; Schilling, M., 319 ; Schlink, W., 49 ; Schmidt, E., 200 ; Schur, F., 49, 367 ; Schwarz, H. A., 200 ; Schwarz, S. A., 150 ; Scott, R. F., 99 ; Severi, F., 98 ; Shaw, W. N., 96 ; Shumway, R. R., 519 ; Silverman, L. L., 471 ; Sinclair J. E., 50 ; Sinclair, M. E., 98 ; Slaught, H. E., 412 ; Slichter, C. S., 520 ; Slobin, H. L., 98 ; Smith, D. E., 265 ; Smith, M. F., 266 ; Somigliana, C., 98 ; Sommerfeld, A., 268 ; Spiess, O., 415 ; Stahl, H. v., 471 ; Stanton, W. F., 50 ; Stéphanos, C., 149 ; Stocker, J. E., 50 ; Stone, W. B., 100 ; Stromquist, C. E., 519 ; Study, E., 149.

Tanner, W. L., 519 ; Taylor, C., 51 ; Thaer, C., 415 ; Thomson, J. J.,

96, 468; Tietze, H., 201; Tikhoff, M., 266; Timerding, H. E., 470; Townley, S. D., 415; Turnbull, N. W., 411; Tyler, H. W., 145, 469.
Underhill, A. L., 519.
Vailati, G., 520; Vanecek, M. N., 98; Van Amringe, J. H., 416; Van Benschoten, A. L., 98; Vogt, W., 470; Volterra, V., 98, 146, 415.
Waldo, C. A., 150; Watson, G. N., 411; Wernicke, P., 99; White, H. S., 95, 145; White, W. F., 268; Whyte, W. A., 150; Wiener, H., 149; Wilson, N. R., 98; Woods, F. S., 519; Wolff, H. C., 98; Worthington, E. R., 98.
Young, J. W. A., 265.
Zoretti, E., 49, 415.

Prizes:
Belgian Academy, 44, 518; Berlin Academy, 46; Berlin University, 46; Göttingen Academy, 44, 45; Göttingen University, 267; Italian Scientific Society, 98; Jablonowski, 518; Lombardy Royal Institute, 45; King Leopold, 469; Paris Academy, 47, 265; Smith, 411; Turin Academy, 317; Wolfskehl, 44.

Universities and Technical Schools:
Berlin, 46, 412.
Bonn, 47, 412.
Braunschweig, 148.
Charlottenburg, 148.
Chicago, 413.
Columbia, 413.
Cornell, 413.
Darmstadt, 148.
Göttingen, 47, 267, 317, 517.
Harvard, 469.
Illinois, 414.
Indiana, 517.
Johns Hopkins, 470.
Leipzig, 47, 412.
Munich, 48, 149, 366.
Paris, 147, 365.
Pennsylvania, 470, 517.
Princeton, 366.
Strassburg, 48, 366.
Stuttgart, 149.
Yale, 414.